Learning by Reading Original Mathematics

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In spite of the large number of good textbooks available today, our undergraduate and graduate students can greatly benefit by regarding the corpus of received original mathematical works as accessible, fascinating, relevant reading material with direct bearing on their coursework. When mathematicians such as Landen, Daniel Bernoulli, or Newton discussed summation of series, difference equations, or areas under curves, they wrote with the passion and enthusiasm that accompanies the discovery of new insights and relationships. By reading these original works, our students may more clearly see mathematics as dynamic and evolving. They can better understand theorems and formulas as arising out of real and significant mathematical questions, with connections to other problems. Moreover, many older works present their results in simple form, without the elaborate techniques developed later. Also, the results in older works may be incomplete or lacking in rigor; this apparent deficiency gives students an opportunity to provide that rigor, or to understand the need for further refinement.

Reflecting a sense of excitement and the new prospects revealed by his discovery of infinite series, in 1670–71 Newton wrote in his *De Methodis* [5]

> I am amazed that it has occurred to no one (if you except N. Mercator with his quadrature of the hyperbola) to fit the doctrine recently established for decimal numbers in similar fashion to variables, especially since the way is then open to more striking consequences. For since this doctrine in species has the same relationship to Algebra that the doctrine in decimal numbers has to common Arithmetic, its operations of Addition, Subtraction, Multiplication, Division and Root-extraction may easily be learnt from the latter's provided the reader be skilled in each, both Arithmetic and Algebra, and appreciate the correspondence between decimal numbers and algebraic terms continued to infinity ... .

Using Newton's analogy between arithmetic and algebra, just as $1/3$ or $\sqrt{3}$ could be written as infinite decimals, $1/(1+x^2)$ and $\sqrt{1+x^2}$ could be expanded in infinite series. As one of the "more striking consequences", he applied the method of successive approximation for solving an algebraic equation $f(y) = 0$, gleaned from his study of Viète, to solve a polynomial equation $f(x, y) = 0$, but using infinite series. Newton showed that such a solution had to be of the form $y = x^\alpha g(x)$, with α rational and $g(x)$ a power series. He thereby obtained the first statement of the implicit function theorem. The factor α was determined by the method called Newton's polygon; interestingly, in his *De Analysi* of 1669, Newton had not yet taken this factor into account, though a simple example shows the need for it. Against the backdrop of Descartes's geometry, Newton's results appear in a meaningful context, motivated by his effort to find the area under a curve defined by $f(x, y) = 0$, a problem he solved by integrating $y = x^\alpha g(x)$ term by term.

Students may be concerned that Newton did not consider convergence questions, and this itself provides ample food for thought. Indeed, the algebraic geometer Abhyankar [1] wrote, "...Newton's proof, being algorithmic, applies equally well to power

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series, whether they converge or not. Moreover, and that is the main point, Newton's algorithmic proof leads to numerous other existence theorems, while Puiseux's existential proof does not do so." In his *Adventures of a Mathematician*, Ulam recalls Banach's remark, "Good mathematicians see analogies between theorems or theories, the very best ones see analogies between analogies." In Newton, one finds analogy being employed by a master.

In the work of Daniel Bernoulli, students may perceive the dynamic development of mathematical ideas, as well as the intellectual growth and joy of discovery within Bernoulli himself. In 1724 Daniel Bernoulli wrote in his *Exercitationes* that there was no formula for the general term of the sequence 1, 3, 4, 7, 11, 18, …, a sequence satisfying the second-order difference equation $a_n + a_{n-1} = a_{n+1}$. His cousin Niklaus I Bernoulli informed him that, in fact, the general term could be expressed as $[(1+\sqrt{5})/2]^n + [(1-\sqrt{5})/2]^n$. On rethinking, Daniel Bernoulli discovered the method for solving linear difference equations usually presented in textbooks. He explained in a 1728 paper [2] that if $x_1, x_2, \ldots, x_m$ were the solutions of the algebraic equation $\sum_{k=0}^{m} \beta_k x^k = 0$, then the general solution of the difference equation

$$\sum_{k=0}^{m} \beta_k a_{n+k} = 0 \quad \text{would be} \quad a_n = \sum_{k=1}^{m} A_k x_k^n$$

where $A_1, A_2, \ldots, A_m$ were arbitrary constants determined by the m initial values. In 1717 de Moivre had published the method of generating functions, or, in his words, recurrent series, to solve difference equations. Bernoulli's novel idea was the construction of the general solution as a linear combination of special solutions, a method so original that it took a decade to extend it to linear differential equations. In fact, Euler wrote to Johann Bernoulli in September 1739 that he was astonished to discover that the solution of a linear differential equation with constant coefficients was determined by an algebraic equation.

Daniel Bernoulli applied his discovery to find the numerically largest solution of an algebraic equation. From the general solution, he easily determined the numerically largest value to be approximately a_{n+1}/a_n, for n sufficiently large. Bernoulli commented that his result, even if not useful, was beautiful. But, in fact, he applied it to obtain the zeros of some Laguerre polynomials arising in his work on the frequencies of oscillation of hanging chains. At the end of his 1728 paper, he solved the difference equation

$$a_{n+1} - 2\cos\theta a_n + a_{n-1} = 0.$$

Surprisingly, this gave him a new derivation of de Moivre's 1707 formula

$$2\cos(n\theta) = (\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n,$$

from which de Moivre [4], in a few lines, had obtained the important and useful formula

$$x^{2n} - 2a^n x^n \cos(n\theta) + a^{2n} = \prod_{k=0}^{n-1} \left(x^2 - 2ax\cos\left(\theta + \frac{2k\pi}{n}\right) + a^2 \right).$$

It would be a fascinating experience for a student to work through the derivation of this factorization formula from a difference equation.

Landen's efforts to sum $\sum_{n=1}^{\infty} 1/n^2$ produced a brilliant and essentially correct idea. But, since complex analysis had not been developed, his solution was incomplete, with many loose ends. Students may study Landen's insights with great benefit, as they attempt to fill in the gaps and render his work more coherent; they may observe that routine calculations can yield results of tremendous significance.

In 1729 Euler discussed the dilogarithm function

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\,dt = \sum_{n=1}^{\infty} \frac{x^n}{n^2},$$

where the last equality would hold for $|x| \le 1$; though he found some interesting results, he was unable to evaluate $\mathrm{Li}_2(1)$ from the integral representation. In a paper of 1760 in the *Philosophical Transactions*, Landen [3] showed that this evaluation required the use of $\log(-1)$. Unaware of Euler's work on logarithms, Landen set out to determine $\log(-1)$ by first differentiating $x = \sin z$ to obtain $dz/\sqrt{-1} = dx/\sqrt{x^2-1}$. Integrating this, he got

$$\frac{z}{\sqrt{-1}} = \log\left(\frac{x + \sqrt{x^2-1}}{\sqrt{-1}}\right).$$

He set $z = \pi/2$ and $x = 1$ so that $\log\sqrt{-1} = -\pi/(2\sqrt{-1})$. Since a square root must have two values, he concluded

$$\log(-1) = \pm\frac{\pi}{\sqrt{-1}}. \tag{1}$$

To evaluate $\mathrm{Li}_2(1)$, he started with the calculation

$$x^{-1} + \frac{x^{-2}}{2} + \frac{x^{-3}}{3} + \cdots = \log\frac{1}{1-1/x} = \log x + \log\frac{1}{1-x} - \frac{\pi}{\sqrt{-1}}, \tag{2}$$

where he chose the negative sign from equation (1). Next, Landen divided (2) by x to obtain

$$-\mathrm{Li}_2(1/x) = -\frac{\pi}{\sqrt{-1}}\log x + \frac{1}{2}(\log x)^2 + \mathrm{Li}_2(x) + C. \tag{3}$$

To find C, he first took $x = 1$ to get $C = -2\sum_{n=1}^{\infty} 1/n^2$ and to evaluate this series, he set $x = -1$ in (3). But this time, he pragmatically chose the positive sign in (1) and arrived at

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

As his computations indicate, formula (3) is correct only for $x \geq 1$. Landen proceeded to divide (3) by x and then integrate; he repeated the process indefinitely. The resulting formulas gave him the values of

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{2k+1}}.$$

Observe that the series $\mathrm{Li}_2(x)$ and $\mathrm{Li}_2(1/x)$ converge when $x = e^{i\theta}$, with $0 < \theta \leq 2\pi$. When this value of x is substituted in (3), the resulting formula is incorrect; choosing the plus sign in (3) makes the formula valid, yielding the Fourier series expansion of the second Bernoulli polynomial! Clearly, a student of complex analysis could learn a great deal by delving into Landen's remarkable summations.

An ever-increasing number of old mathematics papers and books are easily available online and in print, making it very convenient to use such material in class discussions, assignments, or math club activities. For example, Newton's basic analogy may be presented in a calculus lecture, and Newton's polygon assigned as a project. The wealth of old mathematics applicable in our courses is astonishing, including both circumscribed and open-ended problems, and this treasure trove is a wonderful teaching resource.

References

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[4] A. de Moivre, *Miscellanea Analytica*, London, 1730, 1–2.

[5] I. Newton, *Mathematical Papers of Isaac Newton*, vol. 3 (D. T. Whiteside, ed.), Cambridge University Press, Cambridge, 1969, 33–35.