PUTNAM TRAINING PROBLEMS, 2014

(Last updated: August 21, 2014)

REMARK. This is a list of problems discussed during the training sessions of the NU Putnam team and arranged by subjects. The document has three parts, the first one contains the problems, the second one hints, and the solutions are in the third part. —Miguel A. Lerma

EXERCISES

1. **Induction.**

1.1. Prove that $n! > 2^n$ for all $n \geq 4$.

1.2. Prove that for any integer $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

1.3. Let a and b two distinct integers, and n any positive integer. Prove that $a^n - b^n$ is divisible by $a - b$.

1.4. The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ is defined as a sequence whose two first terms are $F_0 = 0$, $F_1 = 1$ and each subsequent term is the sum of the two previous ones: $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). Prove that $F_n < 2^n$ for every $n \geq 0$.

1.5. Let r be a number such that $r + 1/r$ is an integer. Prove that for every positive integer n, $r^n + 1/r^n$ is an integer.

1.6. Find the maximum number $R(n)$ of regions in which the plane can be divided by n straight lines.

1.7. We divide the plane into regions using straight lines. Prove that those regions can be colored with two colors so that no two regions that share a boundary have the same color.

1.8. A great circle is a circle drawn on a sphere that is an "equator", i.e., its center is also the center of the sphere. There are n great circles on a sphere, no three of which meet at any point. They divide the sphere into how many regions?

1.9. We need to put n cents of stamps on an envelop, but we have only (an unlimited supply of) 5¢ and 12¢ stamps. Prove that we can perform the task if $n \geq 44$.

1.10. A chessboard is a 8×8 grid (64 squares arranged in 8 rows and 8 columns), but here we will call "chessboard" any $m \times m$ square grid. We call *defective* a chessboard if one of its squares is missing. Prove that any $2^n \times 2^n$ ($n \geq 1$) defective chessboard can be tiled (completely covered without overlapping) with L-shaped trominos occupying exactly 3 squares, like this  .

1.11. This is a modified version of the game of Nim (in the following we assume that there is an unlimited supply of tokens.) Two players arrange several piles of tokens in a row. By turns each of them takes one token from one of the piles and adds at will as many tokens as he or she wishes to piles placed to the left of the pile from which the token was taken. Assuming that the game ever finishes, the player that takes the last token wins. Prove that, no matter how they play, the game will eventually end after finitely many steps.

1.12. Call an integer square-full if each of its prime factors occurs to a second power (at least). Prove that there are infinitely many pairs of consecutive square-fulls.

1.13. Prove that for every $n \geq 2$, the expansion of $(1+x+x^2)^n$ contains at least one even coefficient.

1.14. We define recursively the *Ulam numbers* by setting $u_1 = 1$, $u_2 = 2$, and for each subsequent integer n, we set n equal to the next Ulam number if it can be written uniquely as the sum of two different Ulam numbers; e.g.: $u_3 = 3$, $u_4 = 4$, $u_5 = 6$, etc. Prove that there are infinitely many Ulam numbers.

1.15. Prove Bernoulli's inequality, which states that if $x > -1$, $x \neq 0$ and n is a positive integer greater than 1, then $(1+x)^n > 1 + nx$.

2. **Inequalities.**

2.1. If $a, b, c > 0$, prove that $(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$.

2.2. Prove that $n! < \left(\dfrac{n+1}{2}\right)^n$, for $n = 2, 3, 4, \ldots,$

2.3. If $0 < p$, $0 < q$, and $p + q < 1$, prove that $(px + qy)^2 \leq px^2 + qy^2$.

2.4. If $a, b, c \geq 0$, prove that $\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$.

2.5. Let $x, y, z > 0$ with $xyz = 1$. Prove that $x + y + z \leq x^2 + y^2 + z^2$.

2.6. Show that

$$\sqrt{a_1^2 + b_1^2} + \sqrt{a_2^2 + b_2^2} + \cdots + \sqrt{a_n^2 + b_n^2} \quad \geq \quad \sqrt{(a_1 + a_2 + \cdots + a_n)^2 + (b_1 + b_2 + \cdots + b_n)^2}$$

2.7. Find the minimum value of the function $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$, where $x_1, x_2, \ldots, x_n$ are positive real numbers such that $x_1 x_2 \cdots x_n = 1$.

2.8. Let $x, y, z \geq 0$ with $xyz = 1$. Find the minimum of

$$S = \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\,.$$

2.9. If $x, y, z > 0$, and $x + y + z = 1$, find the minimum value of

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\,.$$

2.10. Prove that in a triangle with sides a, b, c and opposite angles A, B, C (in radians) the following relation holds:

$$\frac{a\,A + b\,B + c\,C}{a+b+c} \geq \frac{\pi}{3}\,.$$

2.11. (Putnam, 2003) Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq ((a_1+b_1)(a_2+b_2)\cdots(a_n+b_n))^{1/n}$$

2.12. The notation $n!^{(k)}$ means take factorial of n k times. For example, $n!^{(3)}$ means $((n!)!)!$ What is bigger, $1999!^{(2000)}$ or $2000!^{(1999)}$?

2.13. Which is larger, 1999^{1999} or 2000^{1998}?

2.14. Which is larger, $\log_2 3$ or $\log_3 5$?

2.15. Prove that there are no positive integers a, b such that $b^2 + b + 1 = a^2$.

2.16. (Inspired in Putnam 1968, B6) Prove that a polynomial with only real roots and all coefficients equal to ± 1 has degree at most 3.

2.17. (Putnam 1984) Find the minimum value of

$$(u-v)^2 + \left(\sqrt{2-u^2} - \frac{9}{v}\right)^2$$

for $0 < u < \sqrt{2}$ and $v > 0$.

2.18. Show that $\displaystyle \frac{1}{\sqrt{4n}} \leq \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{2n-1}{2n}\right) < \frac{1}{\sqrt{2n}}$.

2.19. (Putnam, 2004) Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

2.20. Let $a_1, a_2, \dots, a_n$ be a sequence of positive numbers, and let $b_1, b_2, \dots, b_n$ be any permutation of the first sequence. Show that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n\,.$$

2.21. (Rearrangement Inequality.) Let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ increasing sequences of real numbers, and let $x_1, x_2, \dots, x_n$ be any permutation of $b_1, b_2, \dots, b_n$. Show that

$$\sum_{i=1}^{n} a_i b_i \geq \sum_{i=1}^{n} a_i x_i \,.$$

2.22. Prove that the p-mean tends to the geometric mean as p approaches zero. In other other words, if $a_1, \dots, a_n$ are positive real numbers, then

$$\lim_{p \to 0} \left(\frac{1}{n} \sum_{k=1}^{n} a_k^p \right)^{1/p} = \left(\prod_{k=1}^{n} a_k \right)^{1/n}$$

2.23. If a, b, and c are the sides of a triangle, prove that

$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c} \geq 3 \,.$$

2.24. Here we use Knuth's up-arrow notation: $a \uparrow b = a^b$, $a \uparrow\uparrow b = \underbrace{a \uparrow (a \uparrow (\dots \uparrow a))}_{b \text{ copies of } a}$, so e.g. $2 \uparrow\uparrow 3 = 2 \uparrow (2 \uparrow 2)) = 2^{2^2}$. What is larger, $2 \uparrow\uparrow 2011$ or $3 \uparrow\uparrow 2010$?

2.25. Prove that $e^{1/e} + e^{1/\pi} \geq 2e^{1/3}$.

2.26. Prove that the function $f(x) = \sum_{i=1}^{n} (x - a_i)^2$ attains its minimum value at $x = \overline{a} = \dfrac{a_1 + \cdots + a_n}{n}$.

2.27. Find the positive solutions of the system of equations

$$x_1 + \frac{1}{x_2} = 4 \,, \quad x_2 + \frac{1}{x_3} = 1 \,, \dots , \quad x_{99} + \frac{1}{x_{100}} = 4 \,, \quad x_{100} + \frac{1}{x_1} = 1 \,.$$

2.28. Prove that if the numbers a, b, and c satisfy the inequalities $|a-b| \geq |c|$, $|b-c| \geq |a|$, $|c-a| \geq |b|$, then one of those numbers is the sum of the other two.

2.29. Find the minimum of $\sin^3 x / \cos x + \cos^3 x / \sin x$, $0 < x < \pi/2$.

2.30. Let $a_i > 0$, $i = 1, \dots, n$, and $s = a_1 + \cdots + a_n$. Prove

$$\frac{a_1}{s-a_1} + \frac{a_2}{s-a_2} + \cdots + \frac{a_n}{s-a_n} \geq \frac{n}{n-1} \,.$$

2.31. Find the maximum value of $f(x) = \sin^4(x) + \cos^4 x$ for $x \in \mathbb{R}$

2.32. Let a, b, c be positive real numbers. Prove that $2\left(\frac{a+b}{2} - \sqrt{ab}\right) \leq 3\left(\frac{a+b+c}{3} - \sqrt[3]{abc}\right)$. When is equality attained?

2.33. (Generalization of previous problem.) Let a_n, $n = 1, 2, 3, \dots$ be a sequence of positive real numbers. Prove that $b_n = a_1 + \cdots + a_n - n\sqrt[n]{a_1 \cdots a_n}$ is an increasing sequence of non-negative real numbers.

3. Number Theory.

3.1. Show that the sum of two consecutive primes is never twice a prime.

3.2. Can the sum of the digits of a square be (a) 3, (b) 1977?

3.3. Prove that there are infinitely many prime numbers of the form $4n + 3$.

3.4. Prove that the fraction $(n^3 + 2n)/(n^4 + 3n^2 + 1)$ is in lowest terms for every possible integer n.

3.5. Let $p(x)$ be a non-constant polynomial such that $p(n)$ is an integer for every positive integer n. Prove that $p(n)$ is composite for infinitely many positive integers n. (This proves that there is no polynomial yielding only prime numbers.)

3.6. Prove that two consecutive Fibonacci numbers are always relatively prime.

3.7. Show that if $a^2 + b^2 = c^2$, then $3|ab$.

3.8. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ can never be an integer for $n \geq 2$.

3.9. Let $f(n)$ denote the sum of the digits of n. Let $N = 4444^{4444}$. Find $f(f(f(N)))$.

3.10. Show that there exist 1999 consecutive numbers, each of which is divisible by the cube of some integer greater than 1.

3.11. Find all triples of positive integers (a, b, c) such that

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) = 2\,.$$

3.12. Find all positive integer solutions to $abc - 2 = a + b + c$.

3.13. (USAMO, 1979) Find all non-negative integral solutions $(n_1, n_2, \dots, n_{14})$ to

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599\,.$$

3.14. The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \dots$ is defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that for some $k > 0$, F_k is a multiple of $10^{10^{10^{10}}}$.

3.15. Do there exist 2 irrational numbers a and b greater than 1 such that $\lfloor a^m \rfloor \neq \lfloor b^n \rfloor$ for every positive integers m, n?

3.16. The numbers 2^{2005} and 5^{2005} are written one after the other (in decimal notation). How many digits are written altogether?

3.17. If p and p^2+2, are primes show that p^3+2 is prime.

3.18. Suppose $n>1$ is an integer. Show that n^4+4^n is not prime.

3.19. Let m and n be positive integers such that $m < \lfloor \sqrt{n} + \frac{1}{2} \rfloor$. Prove that $m + \frac{1}{2} < \sqrt{n}$.

3.20. Prove that the function $f(n) = \lfloor n + \sqrt{n} + 1/2 \rfloor$ $(n = 1, 2, 3, \dots)$ misses exactly the squares.

3.21. Prove that there are no primes in the following infinite sequence of numbers:

$$1001,\ 1001001,\ 1001001001,\ 1001001001001, \dots$$

3.22. (Putnam 1975, A1.) For positive integers n define $d(n) = n - m^2$, where m is the greatest integer with $m^2 \le n$. Given a positive integer b_0, define a sequence b_i by taking $b_{k+1} = b_k + d(b_k)$. For what b_0 do we have b_i constant for sufficiently large i?

3.23. Let $a_n = 10 + n^2$ for $n \ge 1$. For each n, let d_n denote the gcd of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

3.24. Suppose that the positive integers x, y satisfy $2x^2 + x = 3y^2 + y$. Show that $x - y$, $2x + 2y + 1$, $3x + 3y + 1$ are all perfect squares.

3.25. If $2n+1$ and $3n+1$ are both perfect squares, prove that n is divisible by 40.

3.26. How many zeros does 1000! ends with?

3.27. For how many k is the binomial coefficient $\binom{100}{k}$ odd?

3.28. Let n be a positive integer. Suppose that 2^n and 5^n *begin* with the same digit. What is the digit?

3.29. Prove that there are no four consecutive non-zero binomial coefficients $\binom{n}{r}$, $\binom{n}{r+1}$, $\binom{n}{r+2}$, $\binom{n}{r+3}$ in arithmetic progression.

3.30. (Putnam 1995, A1) Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another.

3.31. (Putnam 2003, A1) Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

3.32. (Putnam 2001, B-1) Let n be an even positive integer. Write the numbers $1, 2, \dots, n^2$ in the squares of an $n \times n$ grid so that the kth row, from right to left is

$$(k-1)n+1, (k-1)n+2, \dots, (k-1)n+n\,.$$

Color the squares of the grid so that half the squares in each row and in each column are read and the other half are black (a chalkboard coloring is one possibility). Prove that for each such coloring, the sum of the numbers on the red squares is equal to the sum of the numbers in the black squares.

3.33. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1s and 0s, beginning and ending with 1?

3.34. Prove that if n is an integer greater than 1, then n does not divide $2^n - 1$.

3.35. The digital root of a number is the (single digit) value obtained by repeatedly adding the (base 10) digits of the number, then the digits of the sum, and so on until obtaining a single digit—e.g. the digital root of 65,536 is 7, because $6+5+5+3+6 = 25$ and $2+5 = 7$. Consider the sequence $a_n =$ integer part of $10^n\pi$, i.e.,

$$a_1 = 31\,, \quad a_2 = 314\,, \quad a_3 = 3141\,, \quad a_4 = 31415\,, \quad a_5 = 314159\,, \quad \dots$$

and let b_n be the sequence

$$b_1 = a_1\,, \quad b_2 = a_1^{a_2}\,, \quad b_3 = a_1^{a_2^{a_3}}\,, \quad b_4 = a_1^{a_2^{a_3^{a_4}}}\,, \quad \dots$$

Find the digital root of b_{10^6}.

4. Polynomials.

4.1. Find a polynomial with integral coefficients whose zeros include $\sqrt{2}+\sqrt{5}$.

4.2. Let $p(x)$ be a polynomial with integer coefficients. Assume that $p(a) = p(b) = p(c) = -1$, where a, b, c are three different integers. Prove that $p(x)$ has no integral zeros.

4.3. Prove that the sum

$$\sqrt{1001^2+1} + \sqrt{1002^2+1} + \cdots + \sqrt{2000^2+1}$$

is irrational.

4.4. (USAMO 1975) If $P(x)$ denotes a polynomial of degree n such that $P(k) = k/(k+1)$ for $k = 0, 1, 2, \dots, n$, determine $P(n+1)$.

4.5. (USAMO 1984) The product of two of the four zeros of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Find k.

4.6. Let n be an even positive integer, and let $p(x)$ be an n-degree polynomial such that $p(-k) = p(k)$ for $k = 1, 2, \ldots, n$. Prove that there is a polynomial $q(x)$ such that $p(x) = q(x^2)$.

4.7. Let $p(x)$ be a polynomial with integer coefficients satisfying that $p(0)$ and $p(1)$ are odd. Show that p has no integer zeros.

4.8. (USAMO 1976) If $P(x)$, $Q(x)$, $R(x)$, $S(x)$ are polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x)$$

prove that $x - 1$ is a factor of $P(x)$.

4.9. Let a, b, c distinct integers. Can the polynomial $(x-a)(x-b)(x-c)-1$ be factored into the product of two polynomials with integer coefficients?

4.10. Let $p_1, p_2, \ldots, p_n$ distinct integers and let $f(x)$ be the polynomial of degree n given by

$$f(x) = (x - p_1)(x - p_2) \cdots (x - p_n)\,.$$

Prove that the polynomial

$$g(x) = (f(x))^2 + 1$$

cannot be expressed as the product of two non-constant polynomials with integral coefficients.

4.11. Find the remainder when you divide $x^{81} + x^{49} + x^{25} + x^9 + x$ by $x^3 - x$.

4.12. Does there exist a polynomial $f(x)$ for which $xf(x-1) = (x+1)f(x)$?.

4.13. Is it possible to write the polynomial $f(x) = x^{105} - 9$ as the product of two polynomials of degree less than 105 with integer coefficients?

4.14. Find all prime numbers p that can be written $p = x^4 + 4y^4$, where x, y are positive integers.

4.15. (Canada, 1970) Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial with integral coefficients. Suppose that there exist four distinct integers a, b, c, d with $P(a) = P(b) = P(c) = P(d) = 5$. Prove that there is no integer k with $P(k) = 8$.

4.16. Show that $(1 + x + \cdots + x^n)^2 - x^n$ is the product of two polynomials.

4.17. Let $f(x)$ be a polynomial with real coefficients, and suppose that $f(x) + f'(x) > 0$ for all x. Prove that $f(x) > 0$ for all x.

4.18. Evaluate the following determinant: $\begin{vmatrix} 1 & 1 & 1 & 1 \\ w & x & y & z \\ w^2 & x^2 & y^2 & z^2 \\ w^3 & x^3 & y^3 & z^3 \end{vmatrix}$

4.19. Evaluate the following determinant: $\begin{vmatrix} 1 & 1 & 1 & 1 \\ w & x & y & z \\ w^2 & x^2 & y^2 & z^2 \\ w^4 & x^4 & y^4 & z^4 \end{vmatrix}$

4.20. Do there exist polynomials a, b, c, d such that $1 + xy + x^2y^2 = a(x)b(y) + c(x)d(y)$?

4.21. Determine all polynomials such that $P(0) = 0$ and $P(x^2 + 1) = P(x)^2 + 1$.

4.22. Consider the lines that meet the graph

$$y = 2x^4 + 7x^3 + 3x - 5$$

in four distinct points $P_i = [x_i, y_i]$, $i = 1, 2, 3, 4$. Prove that

$$\frac{x_1 + x_2 + x_3 + x_4}{4}$$

is independent of the line, and compute its value.

4.23. Let k be the smallest positive integer for which there exist distinct integers a, b, c, d, e such that

$$(x - a)(x - b)(x - c)(x - d)(x - e)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers for which this minimum k is achieved.

4.24. Find the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

4.25. (Putnam 1999, A1) Find polynomials $f(x)$, $g(x)$, and $h(x)$ such that

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1\,, & \text{if } x < -1\,, \\ 3x + 2\,, & \text{if } -1 \leq x \leq 0\,, \\ -2x + 2\,, & \text{if } x > 0\,. \end{cases}$$

4.26. Suppose that α, β, and γ are real numbers such that

$$\alpha + \beta + \gamma = 2\,,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 14\,,$$
$$\alpha^3 + \beta^3 + \gamma^3 = 17\,.$$

Find $\alpha\beta\gamma$.

4.27. Prove that $(2 + \sqrt{5})^{1/3} - (-2 + \sqrt{5})^{1/3}$ is rational.

4.28. Two players A and B play the following game. A thinks of a polynomial with non-negative integer coefficients. B must guess the polynomial. B has two shots: she can pick a number and ask A to return the polynomial value there, and then she has another such try. Can B win the game?

4.29. Let $f(x)$ a polynomial with real coefficients, and suppose that $f(x)+f'(x)>0$ for all x. Prove that $f(x)>0$ for all x.

4.30. If $a,b,c>0$, is it possible that each of the polynomials $P(x)=ax^2+bx+c$, $Q(x)=cx^2+ax+b$, $R(x)=bx^2+cx+a$ has two real roots?

4.31. Let $f(x)$ and g(x) be nonzero polynomials with real coefficients such that $f(x^2+x+1)=f(x)g(x)$. Show that $f(x)$ has even degree.

4.32. Prove that there is no polynomial $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$ with integer coefficients and of degree at least 1 with the property that $P(0)$, $P(1)$, $P(2)$,..., are all prime numbers.

5. **Complex Numbers.**

5.1. Let m and n two integers such that each can be expressed as the sum of two perfect squares. Prove that mn has this property as well. For instance $17=4^2+1^2$, $13=2^2+3^2$, and $17\cdot 13=221=14^2+5^2$.

5.2. Prove that $\displaystyle\sum_{k=0}^{n}\sin k=\frac{\sin\frac{n}{2}\sin\frac{n+1}{2}}{\sin\frac{1}{2}}$.

5.3. Show that if z is a complex number such that $z+1/z=2\cos a$, then for any integer n, $z^n+1/z^n=2\cos na$.

5.4. Factor $p(z)=z^5+z+1$.

5.5. Find a close-form expression for $\displaystyle\prod_{k=1}^{n-1}\sin\frac{k\pi}{n}$.

5.6. Consider a regular n-gon which is inscribed in a circle with radius 1. What is the product of the lengths of all $n(n-1)/2$ diagonals of the polygon (this includes the sides of the n-gon).

5.7. (Putnam 1991, B2) Suppose f and g are non-constant, differentiable, real-valued functions on $\mathbb{R}$. Furthermore, suppose that for each pair of real numbers x and y

$$f(x+y)=f(x)\,f(y)-g(x)\,g(y)$$
$$g(x+y)=f(x)\,g(y)+g(x)\,f(y)$$

If $f'(0)=0$ prove that $f(x)^2+g(x)^2=1$ for all x.

5.8. Given a circle of n lights, exactly one of which is initially on, it is permitted to change the state of a bulb provided that one also changes the state of every dth bulb after it (where d is a divisor of n strictly less than n), provided that all n/d

bulbs were originally in the same state as one another. For what values of n is it possible to turn all the bulbs on by making a sequence of moves of this kind?

5.9. Suppose that a, b, u, v are real numbers for which $av - bu = 1$. Prove that $a^2 + b^2 + u^2 + v^2 + au + bv \geq \sqrt{3}$.

6. Generating Functions.

6.1. Prove that for any positive integer n

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1},$$

where $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ (binomial coefficient).

6.2. Prove that for any positive integer n

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

6.3. Prove that for any positive integers $k \leq m, n$,

$$\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{m+n}{k}.$$

6.4. Let F_n be the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$, defined recursively $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that

$$\sum_{n=1}^{\infty} \frac{F_n}{2^n} = 2.$$

6.5. Find a recurrence for the sequence $u_n =$ number of nonnegative solutions of

$$2a + 5b = n.$$

6.6. How many different sequences are there that satisfy all the following conditions:
(a) The items of the sequences are the digits 0–9.
(b) The length of the sequences is 6 (e.g. 061030)
(c) Repetitions are allowed.
(d) The sum of the items is exactly 10 (e.g. 111322).

6.7. (Leningrad Mathematical Olympiad 1991) A finite sequence $a_1, a_2, \ldots, a_n$ is called p-balanced if any sum of the form $a_k + a_{k+p} + a_{k+2p} + \cdots$ is the same for any $k = 1, 2, 3, \ldots, p$. For instance the sequence $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$, $a_5 = 3$, $a_6 = 2$ is 3-balanced because $a_1 + a_4 = 1 + 4 = 5$, $a_2 + a_5 = 2 + 3 = 5$, $a_3 + a_6 = 3 + 2 = 5$. Prove that if a sequence with 50 members is p-balanced for $p = 3, 5, 7, 11, 13, 17$, then all its members are equal zero.

7. **Recurrences.**

7.1. Find the number of subsets of $\{1, 2, \ldots, n\}$ that contain no two consecutive elements of $\{1, 2, \ldots, n\}$.

7.2. Determine the maximum number of regions in the plane that are determined by n "vee"s. A "vee" is two rays which meet at a point. The angle between them is any positive number.

7.3. Define a *domino* to be a 1×2 rectangle. In how many ways can an $n \times 2$ rectangle be tiled by dominoes?

7.4. (Putnam 1996) Define a *selfish* set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets are selfish.

7.5. Let $a_1, a_2, \ldots, a_n$ be an ordered sequence of n distinct objects. A *derangement* of this sequence is a permutation that leaves no object in its original place. For example, if the original sequence is $1, 2, 3, 4$, then $2, 4, 3, 1$ is not a derangement, but $2, 1, 4, 3$ is. Let D_n denote the number of derangements of an n-element sequence. Show that

$$D_n = (n-1)(D_{n-1} + D_{n-2})\,.$$

7.6. Let α, β be two (real or complex) numbers, and define the sequence $a_n = \alpha^n + \beta^n$ $(n = 1, 2, 3, \ldots)$. Assume that a_1 and a_2 are integers. Prove that $2^{\lfloor \frac{n-1}{2} \rfloor} a_n$ is an integer for every $n \geq 1$.

7.7. Suppose that $x_0 = 18$, $x_{n+1} = \dfrac{10x_n}{3} - x_{n-1}$, and that the sequence $\{x_n\}$ converges to some real number. Find x_1.

8. **Calculus.**

8.1. Believe it or not the following function is constant in an interval $[a, b]$. Find that interval and the constant value of the function.

$$f(x) = \sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}\,.$$

8.2. Find the value of the following infinitely nested radical

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}}\,.$$

8.3. (Putnam 1995) Evaluate

$$\sqrt[8]{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \cdots}}}$$

Express your answer in the form $(a + b\sqrt{c})/d$, where a, b, c, d, are integers.

8.4. (Putnam 1992) Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f(\tfrac{1}{n}) = \frac{n^2}{n^2+1}\,, \qquad n = 1, 2, 3, \ldots$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \ldots$.

8.5. Compute $\displaystyle\lim_{n\to\infty} \left\{ \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} \right\}$.

8.6. Compute $\displaystyle\lim_{n\to\infty} \left\{ \prod_{k=1}^{n} \left(1 + \frac{k}{n} \right) \right\}^{1/n}$.

8.7. (Putnam 1997) Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} \right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} \right) dx\,.$$

8.8. (Putnam 1990) Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$ ($n, m = 0, 1, 2, \ldots$)? (In other words, is it possible to find integers n and m such that $\sqrt[3]{n} - \sqrt[3]{m}$ is as close as we wish to $\sqrt{2}$?)

8.9. (Leningrad Mathematical Olympiad, 1988) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with $f(x) \cdot f(f(x)) = 1$ for all $x \in \mathbb{R}$. If $f(1000) = 999$, find $f(500)$.

8.10. Let $f : [0, 1] \to \mathbb{R}$ continuous, and suppose that $f(0) = f(1)$. Show that there is a value $x \in [0, 1998/1999]$ satisfying $f(x) = f(x + 1/1999)$.

8.11. For which real numbers c is $(e^x + e^{-x})/2 \le e^{cx^2}$ for all real x?

8.12. Does there exist a positive sequence a_n such that $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty 1/(n^2 a_n)$ are convergent?

9. **Pigeonhole Principle.**

9.1. Prove that any $(n + 1)$-element subset of $\{1, 2, \ldots, 2n\}$ contains two integers that are relatively prime.

9.2. Prove that if we select $n+1$ numbers from the set $S = \{1, 2, 3, \ldots, 2n\}$, among the numbers selected there are two such that one is a multiple of the other one.

9.3. (Putnam 1978) Let A be any set of 20 distinct integers chosen from the arithmetic progression $\{1, 4, 7, \ldots, 100\}$. Prove that there must be two distinct integers in A whose sum if 104.

9.4. Let A be the set of all 8-digit numbers in base 3 (so they are written with the digits 0,1,2 only), including those with leading zeroes such as 00120010. Prove that given 4 elements from A, two of them must coincide in at least 2 places.

9.5. During a month with 30 days a baseball team plays at least a game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

9.6. (Putnam, 2006-B2.) Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

9.7. (IMO 1972.) Prove that from ten distinct two-digit numbers, one can always choose two disjoint nonempty subsets, so that their elements have the same sum.

9.8. Prove that among any seven real numbers $y_1, \ldots, y_7$, there are two such that

$$0 \leq \frac{y_i - y_j}{1 + y_i y_j} \leq \frac{1}{\sqrt{3}}.$$

9.9. Prove that among five different integers there are always three with sum divisible by 3.

9.10. Prove that there exist an integer n such that the first four digits of 2^n are $2, 0, 0, 9$.

9.11. Prove that every convex polyhedron has at least two faces with the same number of edges.

10. **Telescoping.**

10.1. Prove that $\dfrac{1}{1+\sqrt{2}} + \dfrac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \dfrac{1}{\sqrt{99}+\sqrt{100}} = 9\,.$

10.2. Find a closed form for $\displaystyle\sum_{n=1}^{N} n \cdot n!$

10.3. (Putnam 1984) Express

$$\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$

as a rational number.

10.4. (Putnam 1977) Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

10.5. Evaluate the infinite series: $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1}$.

11. **Symmetries.**

11.1. A spherical, 3-dimensional planet has center at $(0, 0, 0)$ and radius 20. At any point of the surface of this planet, the temperature is $T(x, y, z) = (x + y)^2 + (y - z)^2$ degrees. What is the average temperature of the surface of this planet?

11.2. (Putnam 1980) Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}$.

11.3. Consider the following two-player game. Each player takes turns placing a penny on the surface of a rectangular table. No penny can touch a penny which is already on the table. The table starts out with no pennies. The last player who makes a legal move wins. Does the first player have a winning strategy?

12. **Inclusion-Exclusion.**

12.1. How many positive integers not exceeding 1000 are divisible by 7 or 11?

12.2. Imagine that you are going to give n kids ice-cream cones, one cone per kid, and there are k different flavors available. Assuming that no flavor gets mixed, find the number of ways we can give out the cones using all k flavors.

12.3. Let $a_1, a_2, \ldots, a_n$ an ordered sequence of n distinct objects. A *derangement* of this sequence is a permutation that leaves no object in its original place. For example, if the original sequence is $\{1, 2, 3, 4\}$, then $\{2, 4, 3, 1\}$ is not a derangement, but $\{2, 1, 4, 3\}$ is. Let D_n denote the number of derangements of an n-element sequence. Show that

$$D_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right).$$

13. **Combinatorics and Probability.**

13.1. Prove that the number of subsets of $\{1, 2, \dots, n\}$ with odd cardinality is equal to the number of subsets of even cardinality.

13.2. Find the number of subsets of $\{1, 2, \dots, n\}$ that contain no two consecutive elements of $\{1, 2, \dots, n\}$.

13.3. Peter tosses 25 fair coins and John tosses 20 fair coins. What is the probability that they get the same number of heads?

13.4. From where he stands, one step toward the cliff would send a drunken man over the edge. He takes random steps, either toward or away from the cliff. At any step his probability of taking a step away is p, of a step toward the cliff $1 - p$. Find his chance of escaping the cliff as a function of p.

13.5. Two real numbers X and Y are chosen at random in the interval $(0, 1)$. Compute the probability that the closest integer to X/Y is odd. Express the answer in the form $r + s\pi$, where r and s are rational numbers.

13.6. On the unit circle centered at the origin ($x^2 + y^2 = 1$) we pick three points at random. We cut the circle into three arcs at those points. What is the expected length of the arc containing the point $(1, 0)$?

13.7. In a laboratory a handful of thin 9-inch glass rods had one tip marked with a blue dot and the other with a red. When the laboratory assistant tripped and dropped them onto the concrete floor, many broke into three pieces. For these, what was the average length of the fragment with the blue dot?

13.8. We pick n points at random on a circle. What is the probability that the center of the circle will be in the convex polygon with vertices at those points?

14. **Miscellany.**

14.1. (Putnam 1986) What is the units (i.e., rightmost) digit of $\left\lfloor \dfrac{10^{20000}}{10^{100} + 3} \right\rfloor$?

14.2. (IMO 1975) Prove that there are infinitely many points on the unit circle $x^2 + y^2 = 1$ such that the distance between any two of them is a rational number.

14.3. (Putnam 1988) Prove that if we paint every point of the plane in one of three colors, there will be two points one inch apart with the same color. Is this result necessarily true if we replace "three" by "nine"?

14.4. Imagine an infinite chessboard that contains a positive integer in each square. If the value of each square is equal to the average of its four neighbors to the north, south, west and east, prove that the values in all the squares are equal.

14.5. (Putnam 1990) Consider a paper punch that can be centered at any point of the plane and that, when operated, removes precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?

14.6. (Putnam 1984) Let n be a positive integer, and define

$$f(n) = 1! + 2! + \cdots + n!\,.$$

Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

for all $n \geq 1$.

14.7. (Putnam 1974) Call a set of positive integers "conspiratorial" if no three of them are pairwise relatively prime. What is the largest number of elements in any conspiratorial subset of integers 1 through 16?

14.8. (Putnam 1984) Prove or disprove the following statement: If F is a finite set with two or more elements, then there exists a binary operation $*$ on F such that for all x, y, z in F,

(i) $x * z = y * z$ implies $x = y$ (right cancellation holds), and
(ii) $x * (y * z) \neq (x * y) * z$ (no case of associativity holds).

14.9. (Putnam 1995) For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set S is a collection of disjoint subsets (parts) whose union is S.]

14.10. Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as average of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of distinct elements in A_S?

14.11. Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

14.12. On a table there is a row of fifty coins, of various denominations (the denominations could be of any values). Alice picks a coin from one of the ends and puts it in her pocket, then Bob chooses a coin from one of the ends and puts it in his pocket, and the alternation continues until Bob pockets the last coin. Prove that Alice can play so that she guarantees at least as much money as Bob.

14.13. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \circ f$ has a fixed point, i.e., there is some real number x_0 such that $f(f(x_0)) = x_0$. Prove that f also has a fixed point.

14.14. Prove that $\tan 1^\circ$ is irrational.

14.15. Prove that the integer part of $(5\sqrt{5}+11)^{2n+1}$, $n = 0, 1, 2, \ldots$, is even.

Hints

1.1. —

1.2. For the induction step, rewrite $2^{2(n+1)} - 1$ as a sum of two terms that are divisible by 3.

1.3. For the inductive step assume that step $a^n - b^n$ is divisible by $a - b$ and rewrite $a^{n+1} - b^{n+1}$ as a sum of two terms, one of them involving $a^n - b^n$ and the other one being a multiple of $a - b$.

1.4. Strong induction.

1.5. Rewrite $r^{n+1} + 1/r^{n+1}$ in terms of $r^k + 1/r^k$ with $k \leq n$.

1.6. How many regions can be intersected by the $(n+1)$th line?

1.7. Color a plane divided with n of lines in the desired way, and think how to recolor it after introducing the $(n+1)$th line.

1.8. How many regions can be intersected the by $(n+1)$th circle?

1.9. We have $1 = 5\cdot(-7)+12\cdot 6 = 5\cdot 5+12\cdot(-2)$. Also, prove that if $n = 5x+12y \geq 44$, then either $x \geq 7$ or $y \geq 2$.

1.10. For the inductive step, consider a $2^{n+1} \times 2^{n+1}$ defective chessboard and divide it into four $2^n \times 2^n$ chessboards. One of them is defective. Can the other three be made defective by placing strategically an L?

1.11. Use induction on the number of piles.

1.12. The numbers 8 and 9 form one such pair. Given a pair $(n, n+1)$ of consecutive square-fulls, find some way to build another pair of consecutive square-fulls.

1.13. Look at oddness/evenness of the four lowest degree terms of the expansion.

1.14. Assume that the first m Ulam numbers have already been found, and determine how the next Ulam number (if it exists) can be determined.

1.15. We have $(1+x)^{n+1} = (1+x)^n(1+x)$.

2.1. One way to solve this problem is by using the Arithmetic Mean-Geometric Mean inequality on each factor of the left hand side.

2.2. Apply the Arithmetic Mean-Geometric Mean inequality to the set of numbers $1, 2, \ldots, n$.

2.3. Power means inequality with weights $\frac{p}{p+q}$ and $\frac{q}{p+q}$.

2.4. Power means inequality.

2.5. —

2.6. This problem can be solved by using Minkowski's inequality, but another way to look at it is by an appropriate geometrical interpretation of the terms (as distances between points of the plane.)

2.7. Many minimization or maximization problems are inequalities in disguise. The solution usually consists of "guessing" the maximum or minimum value of the function, and then proving that it is in fact maximum or minimum. In this case, given the symmetry of the function a good guess is $f(1,1,\dots,1)=n$, so try to prove $f(x_1,x_2,\dots,x_n)\geq n$. Use the Arithmetic Mean-Geometric Mean inequality on $x_1,\dots,x_n$.

2.8. Apply the Cauchy-Schwarz inequality to the vectors $(\frac{x}{\sqrt{y+z}},\frac{y}{\sqrt{z+x}},\frac{z}{\sqrt{x+y}})$ and (u,v,w), and choose appropriate values for u,v,w.

2.9. Arithmetic-Harmonic Mean inequality.

2.10. Assume $a\leq b\leq c$, $A\leq B\leq C$, and use Chebyshev's Inequality.

2.11. Divide by the right hand side and use the Arithmetic Mean-Geometric Mean inequality on both terms of the left.

2.12. Note that $n!$ is increasing ($n<m \implies n!<m!$)

2.13. Look at the function $f(x)=(1999-x)\ln(1999+x)$.

2.14. Use the definition of logarithm.

2.15. The numbers b^2 and $(b+1)^2$ are consecutive squares.

2.16. Use the Arithmetic Mean-Geometric Mean inequality on the squares of the roots of the polynomial.

2.17. Think geometrically. Interpret the given expression as the square of the distance between two points in the plane. The problem becomes that of finding the minimum distance between two curves.

2.18. Consider the expressions $P=\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{2n-1}{2n}\right)$ and $Q=\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\cdots\left(\frac{2n-2}{2n-1}\right)$. Note that $\frac{k-1}{k}<\frac{k}{k+1}$, for $k=1,2,\dots$.

2.19. Look at the binomial expansion of $(m+n)^{m+n}$.

2.20. Arithmetic Mean-Geometric Mean inequality.

2.21. Try first the cases $n=1$ and $n=2$. Then use induction.

2.22. Take logarithms and use L'Hôpital.

2.23. Set $x = b + c - a$, $y = c + a - b$, $z = a + b - c$.

2.24. We have $2^{2^2} = 16 < 27 = 3^3$.

2.25. Show that $f(x) = e^{1/x}$ for $x > 0$ is decreasing and convex.

2.26. Prove that $f(x) - f(\overline{a}) \geq 0$.

2.27. By the AM-GM inequality we have $x_1 + \frac{1}{x_2} \geq 2\sqrt{\frac{x_1}{x_2}}, \dots$ Try to prove that those inequalities are actually equalities.

2.28. Square both sides of those inequalities.

2.29. Rearrangement inequality.

2.30. Rearrangement inequality.

2.31. Note that $|\sin x| \leq 1$, so what which is smaller, $\sin^2 x$ or $\sin^4 x$? (Same with $\cos x$.)

2.32. Arithmetic Mean-Geometric Mean inequality.

2.33. Arithmetic Mean-Geometric Mean inequality.

3.1. Contradiction.

3.2. If s is the sum of the digits of a number n, then $n - s$ is divisible by 9.

3.3. Assume that there are finitely many primes of the form $4n+3$, call P their product, and try to obtain a contradiction similar to the one in Euclid's proof of the infinitude of primes.

3.4. Prove that $n^3 + 2n$ and $n^4 + 3n^2 + 1$ are relatively prime.

3.5. Prove that $p(k)$ divides $p(p(k) + k)$.

3.6. Induction.

3.7. Study the equation modulo 3.

3.8. Call the sum S and find the maximum power of 2 dividing each side of the equality

$$n!S = \sum_{k=1}^{n} \frac{n!}{k}\,.$$

3.9. $f(n) \equiv n \pmod 9$.

3.10. Chinese Remainder Theorem.

3.11. The minimum of a, b, c cannot be very large.

3.12. Try changing variables $x = a + 1$, $y = b + 1$, $z = c + 1$.

3.13. Study the equation modulo 16.

3.14. Use the Pigeonhole Principle to prove that the sequence of pairs (F_n, F_{n+1}) is eventually periodic modulo $N = 10^{10^{10^{10}}}$.

3.15. Try $a = \sqrt{6}$, $b = \sqrt{3}$.

3.16. —

3.17. If p is an odd number not divisible by 3, then $p^2 \equiv \pm 1 \pmod 6$.

3.18. Sophie Germain's identity: $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$.

3.19. The number $\sqrt{n}$ is irrational or an integer.

3.20. If $m \neq \lfloor n + \sqrt{n} + 1/2 \rfloor$, what can we say about m?

3.21. Each of the given numbers can be written $p_n(10^3)$, where $p_n(x) = 1+x+x^2+\cdots+x^n$, $n = 1, 2, 3, \dots$.

3.22. Study the cases b_k = perfect square, and b_k = not a perfect square. What can we deduce about b_{k+1} being or not being a perfect square in each case?

3.23. $\gcd(a, b) = \gcd(a, b - a)$.

3.24. What is $(x - y)(2x + 2y + 1)$ and $(x - y)(3x + 3y + 1)$?

3.25. Think modulo 5 and modulo 8.

3.26. Think of 1000! as a product of prime factors and count the number of 2's and the number of 5's in it.

3.27. Find the exponent of 2 in the prime factorization of $\binom{n}{k}$.

3.28. If N begins with digit a then $a \cdot 10^k \leq N < (a + 1) \cdot 10^k$.

3.29. The desired sequence of binomial numbers must have a constant difference.

3.30. Induction. The base case is $1 = 2^0 3^0$. The induction step depends on the parity of n. If n is even, divide by 2. If it is odd, subtract a suitable power of 3.

3.31. If $0 < k \leq n$, is there any such sum with exactly k terms? How many?

3.32. Interpret the grid as a 'sum' of two grids, one with the terms of the form $(k - 1)n$, and the other one with the terms of the form $1, \dots, n$.

3.33. Each of the given numbers can be written $p_n(10^2)$, where $p_n(x) = 1+x+x^2+\cdots+x^n$, $n = 1, 2, 3, \ldots$.

3.34. If n is prime Fermat's Little Theorem yields the result. Otherwise let p be the smallest prime divisor of n...

3.35. The digital root of a number is its reminder modulo 9. Then show that a_1^n ($n = 1, 2, 3, \cdots$) modulo 9 is periodic.

4.1. Call $x = \sqrt{2} + \sqrt{5}$ and eliminate the radicals.

4.2. Factor $p(x) + 1$.

4.3. Prove that the sum is the root of a monic polynomial but not an integer.

4.4. Look at the polynomial $Q(x) = (x + 1)P(x) - x$.

4.5. Use the relationship between zeros and coefficients of a polynomial.

4.6. The $(n - 1)$-degree polynomial $p(x) - p(-x)$ vanishes at n different points.

4.7. For each integer k study the parity of $p(k)$ depending on the parity of k.

4.8. We must prove that $P(1) = 0$. See what happens by replacing x with fifth roots of unity.

4.9. Assume $(x-a)(x-b)(x-c)-1 = p(x)q(x)$, and look at the possible values of $p(x)$ and $q(x)$ for $x = a, b, c$.

4.10. Assume $g(x) = h(x)k(x)$, where $h(x)$ and $k(x)$ are non-constant polynomials with integral coefficients. Prove that the can be assumed to be positive for every x and $h(p_i) = k(p_i) = 1$, $i = 1, \ldots, n$. Deduce that both are of degree n and determine their form. Get a contradiction by equating coefficients in $g(x)$ and $h(x)k(x)$.

4.11. The remainder will be a second degree polynomial. Plug the roots of $x^3 - x$.

4.12. Find the value of $f(n)$ for n integer.

4.13. Assume $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ have integral coefficients and degree less than 105. Look at the product of the roots of $g(x)$

4.14. Sophie Germain's Identity.

4.15. We have that a, b, c, d are distinct roots of $P(x) - 5$.

4.16. One way to solve this problem is by letting $A_{n-1} = 1 + x + \cdots + x^{n-1}$ and doing some algebra.

4.17. Study the behavior of $f(x)$ as $x \to \pm\infty$. Also determine the number of roots of $f(x)$.

4.18. Expand the determinant along the last column and find its zeros as a polynomial in z.

4.19. Expand the determinant along the last column and find its zeros as a polynomial in z.

4.20. Write the given condition in matrix form and give each of x and y three different values.

4.21. Find some polynomial that coincides with $P(x)$ for infinitely many values of x.

4.22. Find intersection points solving a system of equations.

4.23. The numbers a, b, c, d, e are the roots of the given polynomial. How are the roots of a fifth-degree polynomial with exactly 1,2,... non-zero coefficients?

4.24. Find first the set of x verifying the constrain.

4.25. Try with first degree polynomials. Some of those polynomials must change sign precisely at $x = -1$ and $x = 0$. Recall that $|u| = \pm u$ depending on whether $u \geq 0$ or $u < 0$.

4.26. Write the given sums of powers as functions of the elementary symmetric polynomials of α, β, γ.

4.27. Find a polynomial with integer coefficients with that number as one of its roots.

4.28. What happens if B has an upper bound for the coefficients of the polynomials?

4.29. How could $f(x)$ become zero, and how many times? From the behavior of $f(x) + f'(x)$, what can we conclude about the leading coefficient and degree of $f(x)$?

4.30. What conditions must the coefficients satisfy for a second degree polynomial to have two real roots?

4.31. Prove that f(x) cannot have real roots.

4.32. We have that $a_0 = P(0)$ must be a prime number.

5.1. If $m = a^2 + b^2$ and $n = c^2 + d^2$, then consider the product $z = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

5.2. The left hand side of the equality is the imaginary part of $\sum_{k=0}^{n} e^{ik}$.

5.3. What are the possible values of z?

5.4. If $\omega = e^{2\pi i/3}$ then ω and ω^2 are two roots of $p(z)$.

5.5. Write $\sin t = (e^{ti} - e^{-ti})/2i$.

5.6. Assume the vertices of the n-gon placed on the complex plane at the nth roots of unity.

5.7. Look at the function $h(x) = f(x) + ig(x)$.

5.8. Assume the lights placed on the complex plane at the nth roots of unity $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$, where $\zeta = e^{2\pi i/n}$.

5.9. Hint Let $z_1 = a - bi$, $z_2 = u + vi$. We have $|z_1|^2 = a^2 + b^2$, $|z_2| = u^2 + v^2$, $\Re(z_1 z_2) = au + bv$, $\Im(z_1 z_2) = 1$, and must prove $|z_1|^2 + |z_2|^2 + \Re(z_1 z_2) \geq \sqrt{3}$.

6.1. Expand and differentiate $(1 + x)^n$.

6.2. Expand both sides of $(1 + x)^n (1 + x)^n = (1 + x)^{2n}$ and look at the coefficient of x^n.

6.3. Expand both sides of $(1 + x)^m (1 + x)^n = (1 + x)^{m+n}$ and look at the coefficient of x^j.

6.4. Look at the generating function of the Fibonacci sequence.

6.5. Find the generating function of the sequence u_n = number of nonnegative solutions of $2a + 5b = n$.

6.6. The answer equals the coefficient of x^{10} in the expansion of $(1 + x + x^2 + \cdots + x^9)^6$, but that coefficient is very hard to find directly. Try some simplification.

6.7. Look at the polynomial $P(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_{50} x^{49}$, and at its values at 3rd, 5th,... roots of unity.

7.1. The subsets of $\{1, 2, \dots, n\}$ that contain no two consecutive elements can be divided into two classes, the ones not containing n, and the ones containing n.

7.2. The $(n + 1)$th "vee" divides the existing regions into how many further regions?

7.3. The tilings of a $n \times 2$ rectangle by dominoes can be divided into two classes depending on whether we place the rightmost domino vertically or horizontally.

7.4. The minimal selfish subsets of $\{1, 2, \dots, n\}$ can be divided into two classes depending on whether they contain n or not.

7.5. Assume that $b_1, b_2, \dots, b_n$ is a derangement of the sequence $a_1, a_2, \dots, a_n$. How many possible values can b_n have? Once we have fixed the value of b_n, divide the possible derangements into two appropriate classes.

7.6. Find a recurrence for a_n.

7.7. Find a general solution to the recurrence and determine for which value(s) of x_1 the sequence converges.

8.1. $\sqrt{u^2} = |u|$.

8.2. Find the limit of the sequence $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ $(n \geq 1)$.

8.3. Call the limit L. Find some equation verified by L.

8.4. Justify that the desired derivatives must coincide with those of the function $g(x) = 1/(1 + x^2)$.

8.5. Compare the sum to some integral of the form $\int_a^b \frac{1}{x}\,dx$.

8.6. Take logarithms. Interpret the resulting expression as a Riemann sum.

8.7. Interpret the first series is as a Maclaurin series. Interchange integration and summation with the second series (don't forget to justify why the interchange is "legitimate".)

8.8. In fact *any* real number r is the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$. We want $r \approx \sqrt[3]{n} - \sqrt[3]{m}$, i.e., $r + \sqrt[3]{m} \approx \sqrt[3]{n}$. Note that $\sqrt[3]{n+1} - \sqrt[3]{n} \to 0$ as $n \to \infty$.

8.9. If $y \in f(\mathbb{R})$ what is $f(y)$?

8.10. Consider the function $g(x) = f(x) - f(x + 1/999)$. Use the intermediate value theorem.

8.11. Compare Taylor expansions.

8.12. If they were convergent their sum would be convergent too.

9.1. Divide the set into n subsets each of which has only pairwise relatively prime numbers.

9.2. Divide the set into n subsets each of which contains only numbers which are multiple or divisor of the other ones.

9.3. Look at pairs of numbers in that sequence whose sum is precisely 104. Those pairs may not cover the whole progression, but that can be fixed…

9.4. Prove that for each $k = 1, 2, \dots, 8$, at least 2 of the elements given coincide at place k. Consider a pair of elements which coincide at place 1, another pair of elements which coincide at place 2, and so on. How many pairs of elements do we have?

9.5. Consider the sequences a_i = number of games played from the 1st through the jth day of the month, and $b_j = a_j + 14$. Put them together and use the pigeonhole principle to prove that two elements must be the equal.

9.6. Consider the fractional part of sums of the form $s_i = x_1 + \cdots + x_i$.

9.7. Consider the number of different subsets of a ten-element set, and the possible number of sums of at most ten two-digit numbers.

9.8. Write $y_i = \tan x_i$, with $-\frac{\pi}{2} \leq x_i \leq \frac{\pi}{2}$ $(i = 1, \dots, 7)$. Find appropriate "boxes" for the x_is in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$

9.9. Classify the numbers by their reminder when divided by 3.

9.10. We must prove that there are positive integers n, k such that

$$2009 \cdot 10^k \leq 2^n < 2010 \cdot 10^k \,.$$

9.11. Look at the face with the maximum number of edges and its neighbors.

10.1. Rationalize and telescope.

10.2. Write $n = (n + 1) - 1$.

10.3. Try to re-write the nth term of the sum as $\frac{A_k}{3^k - 2^k} - \frac{B_k}{3^{k+1} - 2^{k+1}}$.

10.4. If you write a few terms of the product you will notice a lot of cancellations. Factor the numerator and denominator of the nth term of the product and cancel all possible factors from $k = 2$ to $k = N$. You get an expression in N. Find its limit as $N \to \infty$.

10.5. Write the nth term as a sum of two partial fractions.

11.1. Start by symmetrizing the given function:

$$f(x, y, z) = T(x, y, z) + T(y, z, x) + T(z, x, y) \,.$$

11.2. Look at the expression $f(x) + f(\frac{\pi}{2} - x)$.

11.3. What kind of symmetry can the first player take advantage of?

12.1. —

12.2. Find the number of distributions of ice-cream cones without the restriction "using all k flavors". Then remove the distributions in which at least one of the flavors is unused.

12.3. If P_i is the set of permutations fixing element a_i, then the set of non-derangements are the elements of the $P_1 \cup P_2 \cup \cdots \cup P_n$.

13.1. Find the numbers and subtract. Or find a bijection between the subsets with odd cardinality and those with even cardinality.

13.2. Find a bijection between the k-element subsets of $\{1, 2, \dots, n\}$ with no consecutive elements and all k-element subsets of $\{1, 2, \dots, n-k+1\}$.

13.3. The probability of John getting n heads is the same as that of he getting n tails.

13.4. Consider what happens after the first step, and in which ways the man can reach the edge from there.

13.5. Look at the area of the set of points verifying the condition.

13.6. The lengths of the three arcs have identical distributions.

13.7. The lengths of the three pieces have identical distributions.

13.8. Find the probability of the polygon *not* containing the center of the circle.

14.1. Compare to $\dfrac{10^{20000} - 3^{200}}{10^{100} + 3}$.

14.2. If $\cos u$, $\sin u$, $\cos v$, and $\sin v$ are rational, so are $\cos(u+v)$ and $\sin(u+v)$.

14.3. Contradiction.

14.4. Since the values are positive integers, one of them must be the smallest one. What are the values of the neighbors of a square with minimum value?

14.5. Try punches at $(0,0)$, $(\pm\alpha, 0)$, … for some appropriate α.

14.6. Note that $\frac{f(n+2)-f(n+1)}{f(n+1)-f(n)}$ is a very simple polynomial in n.

14.7. Start by finding some subset T of S as large as possible and such that any three elements of it are pairwise relatively prime.

14.8. Try a binary operation that depends only on the first element: $x * y = \phi(x)$.

14.9. How many different values of $\pi(x)$ are possible?

14.10. Find a set S attaining the minimum cardinality for A_S.

14.11. Group the terms of the sequence appropriately.

14.12. Is it possible for Alice to force Bob into taking coins only from odd-numbered or even-numbered positions?

14.13. If f has not fixed point then $f(x) - x$ is never zero, and f being continous, $f(x) - x$ will have the same sign for every x.

14.14. Angle addition formulas and $\tan 30° = \frac{1}{\sqrt{3}}$.

14.15. Look at $(5\sqrt{5} + 11)^{2n+1} - (5\sqrt{5} - 11)^{2n+1}$.

Solutions

1.1. We prove it by induction. The basis step corresponds to $n = 4$, and in this case certainly we have $4! > 2^4$ ($24 > 16$). Next, for the induction step, assume the inequality holds for some value of $n \geq 4$, i.e., we assume $n! > 2^n$, and look at what happens for $n + 1$:

$$(n+1)! = n!\,(n+1) \underbrace{>}_{\text{by induction hypothesis}} 2^n(n+1) > 2^n \cdot 2 = 2^{n+1}\,.$$

Hence the inequality also holds for $n + 1$. Consequently it holds for every $n \geq 4$.

1.2. For the basis step, we have that for $n = 1$ indeed $2^{2\cdot 1} - 1 = 4 - 1 = 3$ is divisible by 3. Next, for the inductive step, assume that $n \geq 1$ and $2^{2n} - 1$ is divisible by 3. We must prove that $2^{2(n+1)} - 1$ is also divisible by 3. We have

$$2^{2(n+1)} - 1 = 2^{2n+2} - 1 = 4 \cdot 2^{2n} - 1 = 3 \cdot 2^{2n} + (2^{2n} - 1)\,.$$

In the last expression the last term is divisible by 3 by induction hypothesis, and the first term is also a multiple of 3, so the whole expression is divisible by 3 and we are done.

1.3. By induction. For $n = 1$ we have that $a^1 - b^1 = a - b$ is indeed divisible by $a - b$. Next, for the inductive step, assume that $a^n - b^n$ is divisible by $a - b$. We must prove that $a^{n+1} - b^{n+1}$ is also divisible by $a - b$. In fact:

$$a^{n+1} - b^{n+1} = (a - b)\,a^n + b\,(a^n - b^n)\,.$$

On the right hand side the first term is a multiple of $a - b$, and the second term is divisible by $a - b$ by induction hypothesis, so the whole expression is divisible by $a - b$.

1.4. We prove it by strong induction. First we notice that the result is true for $n = 0$ ($F_0 = 0 < 1 = 2^0$), and $n = 1$ ($F_1 = 1 < 2 = 2^1$). Next, for the inductive step, assume that $n \geq 1$ and assume that the claim is true, i.e. $F_k < 2^k$, for every k such that $0 \leq k \leq n$. Then we must prove that the result is also true for $n + 1$. In fact:

$$F_{n+1} = F_n + F_{n-1} \underbrace{<}_{\text{by induction hypothesis}} 2^n + 2^{n-1} < 2^n + 2^n = 2^{n+1}\,,$$

and we are done.

1.5. We prove it by induction. For $n = 1$ the expression is indeed an integer. For $n = 2$ we have that $r^2 + 1/r^2 = (r + 1/r)^2 - 2$ is also an integer. Next assume that $n > 2$ and that the expression is an integer for $n - 1$ and n. Then we have

$$\left(r^{n+1} + \frac{1}{r^{n+1}}\right) = \left(r^n + \frac{1}{r^n}\right)\left(r + \frac{1}{r}\right) - \left(r^{n-1} + \frac{1}{r^{n-1}}\right)\,,$$

hence the expression is also an integer for $n + 1$.

1.6. By experimentation we easily find:

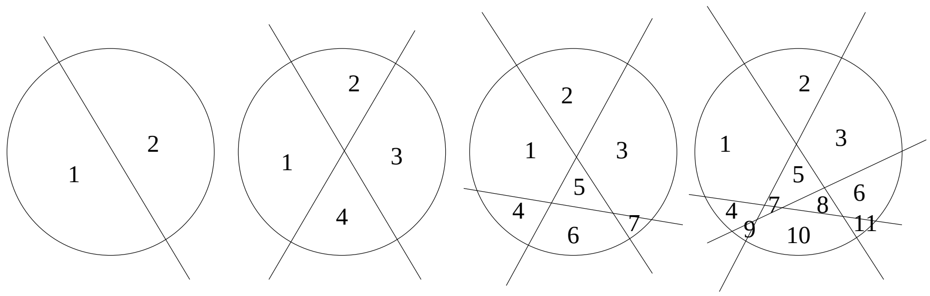


FIGURE 1. Plane regions.

n	1	2	3	4	...
$R(n)$	2	4	7	11	...

A formula that fits the first few cases is $R(n) = (n^2 + n + 2)/2$. We will prove by induction that it works for all $n \geq 1$. For $n = 1$ we have $R(1) = 2 = (1^2 + 1 + 2)/2$, which is correct. Next assume that the property is true for some positive integer n, i.e.:

$$R(n) = \frac{n^2 + n + 2}{2} .$$

We must prove that it is also true for $n + 1$, i.e.,

$$R(n+1) = \frac{(n+1)^2 + (n+1) + 2}{2} = \frac{n^2 + 3n + 4}{2} .$$

So lets look at what happens when we introduce the $(n + 1)$th straight line. In general this line will intersect the other n lines in n different intersection points, and it will be divided into $n + 1$ segments by those intersection points. Each of those $n + 1$ segments divides a previous region into two regions, so the number of regions increases by $n + 1$. Hence:

$$R(n+1) = S(n) + n + 1 .$$

But by induction hypothesis, $R(n) = (n^2 + n + 2)/2$, hence:

$$R(n+1) = \frac{n^2 + n + 2}{2} + n + 1 = \frac{n^2 + 3n + 4}{2} .$$

QED.

1.7. We prove it by induction in the number n of lines. For $n = 1$ we will have two regions, and we can color them with just two colors, say one in red and the other one in blue. Next assume that the regions obtained after dividing the plane with n lines can always be colored with two colors, red and blue, so that no two regions that share a boundary have the same color. We need to prove that such kind of coloring is also possible after dividing the plane with $n+1$ lines. So assume that the plane divided by n lines has been colored in the desired way. After we introduce the

$(n+1)$th line we need to recolor the plane to make sure that the new coloring still verifies that no two regions that share a boundary have the same color. We do it in the following way. The $(n+1)$th line divides the plane into two half-planes. We keep intact the colors in all the regions that lie in one half-plane, and reverse the colors (change red to blue and blue to red) in all the regions of the other half-plane. So if two regions share a boundary and both lie in the same half-plane, they will still have different colors. Otherwise, if they share a boundary but are in opposite half-planes, then they are separated by the $(n+1)$th line; which means they were part of the same region, so had the same color, and must have acquired different colors after recoloring.

1.8. The answer is $f(n) = n^2 - n + 2$. The proof is by induction. For $n = 1$ we get $f(1) = 2$, which is indeed correct. Then we must prove that if $f(n) = n^2 - n + 2$ then $f(n+1) = (n+1)^2 - (n+1) + 2$. In fact, the $(n+1)$th great circle meets each of the other great circles in two points each, so $2n$ points in total, which divide the circle into $2n$ arcs. Each of these arcs divides a region into two, so the number of regions grow by $2n$ after introducing the $(n+1)$th circle. Consequently $f(n+1) = n^2 - n + 2 + 2n = n^2 + n + 2 = (n+1)^2 - (n+1) + 2$, QED.

1.9. We proceed by induction. For the basis step, i.e. $n = 44$, we can use four 5¢ stamps and two 12¢ stamps, so that $5 \cdot 4 + 12 \cdot 2 = 44$. Next, for the induction step, assume that for a given $n \geq 44$ the task is possible by using x 5¢ stamps and y 12¢ stamps, i.e, $n = 5x + 12y$. We must now prove that we can find some combination of x' 5¢ stamps and y' 12¢ stamps so that $n+1 = 5x' + 12y'$. First note that either $x \geq 7$ or $y \geq 2$ — otherwise we would have $x \leq 6$ and $y \leq 1$, hence $n \leq 5 \cdot 6 + 12 \cdot 1 = 42 < 44$, contradicting the hypothesis that $n \geq 44$. So we consider the two cases:

1. If $x \geq 7$, then we can accomplish the goal by setting $x' = x - 7$ and $y' = y + 6$:
$$5x' + 12y' = 5(x-7) + 12(y+6) = 5x + 12y + 1 = n + 1\,.$$
2. On the other hand, if $y \geq 2$ then, we can do it by setting $x' = x + 5$ and $y' = y - 2$:
$$5x' + 12y' = 5(x+5) + 12(y-2) = 5x + 12y + 1 = n + 1\,.$$

1.10. We prove it by induction on n. For $n = 1$ the defective chessboard consists of just a single L and the tiling is trivial. Next, for the inductive step, assume that a $2^n \times 2^n$ defective chessboard can be tiled with L's. Now, given a $2^{n+1} \times 2^{n+1}$ defective chessboard, we can divide it into four $2^n \times 2^n$ chessboards as shown in the figure. One of them will have a square missing and will be defective, so it can be tiled with L's. Then we place an L covering exactly one corner of each of the other $2^n \times 2^n$ chessboards (see figure). The remaining part of each of those chessboards is like a defective chessboard and can be tiled in the desired way too. So the whole $2^{n+1} \times 2^{n+1}$ defective chessboard can be tiled with L's.

1.11. We use induction on the number n of piles. For $n = 1$ we have only one pile, and since each player must take at least one token from that pile, the number of tokes in

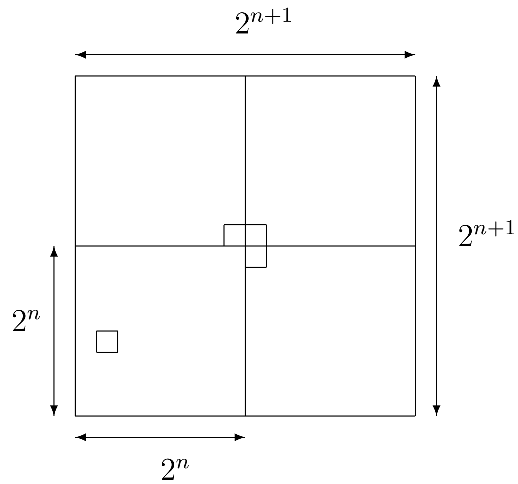


FIGURE 2. A $2^{n+1} \times 2^{n+1}$ defective chessboard.

it will decrease at each move until it is empty. Next, for the induction step, assume that the game with n piles must end eventually. We will prove that the same is true for $n+1$ piles. First note that the players cannot keep taking tokens only from the first n piles, since by induction hypothesis the game with n piles eventually ends. So sooner or later one player must take a token from the $(n+1)$th pile. It does not matter how many tokes he or she adds to the other n piles after that, it is still true that the players cannot keep taking tokens only from the first n piles forever, so eventually someone will take another token from the $(n+1)$th pile. Consequently, the number of tokens in that pile will continue decreasing until it is empty. After that we will have only n piles left, and by induction hypothesis the game will end in finitely many steps after that.[1]

1.12. The numbers 8 and 9 are a pair of consecutive square-fulls. Next, if n and $n+1$ are square-full, so are $4n(n+1)$ and $4n(n+1)+1=(2n+1)^2$.

1.13. For $n = 2, 3, 4, 5, 6$ we have:

$$(1+x+x^2)^2 = 1 + 2x + 3x^2 + 2x^3 + x^4$$
$$(1+x+x^2)^3 = 1 + 3x + 6x^2 + 7x^3 + \cdots$$
$$(1+x+x^2)^4 = 1 + 4x + 10x^2 + 16x^3 + \cdots$$
$$(1+x+x^2)^5 = 1 + 5x + 15x^2 + 30x^3 + \cdots$$
$$(1+x+x^2)^6 = 1 + 6x + 21x^2 + 50x^3 + \cdots$$

[1]An alternate proof based on properties of ordinal numbers is as follows (requires some advanced set-theoretical knowledge.) Here ω = first infinite ordinal number, i.e., the first ordinal after the sequence of natural numbers $0, 1, 2, 3, \ldots$. Let the ordinal number $\alpha = a_0 + a_1\omega + a_2\omega^2 + \cdots + a_{n-1}\omega^{n-1}$ represent a configuration of n piles with $a_0, a_1, \ldots, a_{n-1}$ tokens respectively (read from left to right.) After a move the ordinal number representing the configuration of tokens always decreases. Every decreasing sequence of ordinals numbers is finite. Hence the result.

In general, if $(1+x+x^2)^n = a + bx + cx^2 + dx^3 + \cdots$, then

$$(1+x+x^2)^{n+1} = a + (a+b)x + (a+b+c)x^2 + (b+c+d)x^3 + \cdots ,$$

hence the first four coefficients of $(1+x+x^2)^{n+1}$ depend only on the first four coefficients of $(1+x+x^2)^n$. The same is true is we write the coefficients modulo 2, i.e., as "0" if they are even, or "1" if they are odd. So, if we call $q_n(x) = (1+x+x^2)^n$ with the coefficients written modulo 2, we have

$$\begin{aligned}
q_1(x) &= 1 + 1x + 1x^2 \\
q_2(x) &= 1 + 0x + 1x^2 + 0x^3 + 1x^4 \\
q_3(x) &= 1 + 1x + 0x^2 + 1x^3 + \cdots \\
q_4(x) &= 1 + 0x + 0x^2 + 0x^3 + \cdots \\
q_5(x) &= 1 + 1x + 1x^2 + 0x^3 + \cdots \\
q_6(x) &= 1 + 0x + 1x^2 + 0x^3 + \cdots
\end{aligned}$$

We notice that the first four coefficients of $q_6(x)$ coincide with those of $q_2(x)$, and since these first four coefficients determine the first four coefficients of each subsequent polynomial of the sequence, they will repeat periodically so that those of $q_n(x)$ will always coincide with those of q_{n+4}. Since for $n = 2, 3, 4, 5$ at least one of the first four coefficients of $q_n(x)$ is 0 (equivalently, at least one of the first four coefficients of $(1+x+x^2)^n$ is even), the same will hold for all subsequent values of n.

1.14. Let $U_m = \{u_1, u_2, \dots, u_m\}$ $(m \geq 2)$ be the first m Ulam numbers (written in increasing order). Let S_m be the set of integers greater than u_m that can be written uniquely as the sum of two different Ulam numbers from U_m. The next Ulam number u_{m+1} is precisely the minimum element of S_m, unless S_m is empty, but it is not because $u_{m-1} + u_m \in S_m$.

1.15. By induction. For the base case $n = 2$ the inequality is $(1+x)^2 > 1 + 2x$, obviously true because $(1+x)^2 - (1+2x) = x^2 > 0$. For the induction step, assume that the inequality is true for n, i.e., $(1+x)^n > 1 + nx$. Then, for $n+1$ we have

$$(1+x)^{n+1} = (1+x)^n(1+x) > (1+nx)(1+x) = \\ 1 + (n+1)x + x^2 > 1 + (n+1)x \,,$$

and the inequality is also true for $n+1$.

2.1. Using the Arithmetic Mean-Geometric Mean Inequality on each factor of the LHS we get

$$\left(\frac{a^2b + b^2c + c^2a}{3}\right)\left(\frac{ab^2 + bc^2 + ca^2}{3}\right) \geq \left(\sqrt[3]{a^3b^3c^3}\right)\left(\sqrt[3]{a^3b^3c^3}\right) = a^2b^2c^2 \,.$$

Multiplying by 9 we get the desired inequality.

Another solution consists of using the Cauchy-Schwarz inequality:

$$\begin{aligned}
&(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) = \\
&\qquad \Big((a\sqrt{b})^2 + (b\sqrt{c})^2 + (c\sqrt{a})^2\Big)\Big((\sqrt{b}\,c)^2 + (\sqrt{c}\,a)^2 + (\sqrt{a}\,b)^2\Big) \\
&\qquad \ge (abc + abc + abc)^2 \\
&\qquad = 9a^2b^2c^2 .
\end{aligned}$$

2.2. This result is the Arithmetic Mean-Geometric Mean applied to the set of numbers $1, 2, \dots, n$:

$$\sqrt[n]{1 \cdot 2 \cdot \dots \cdot n} < \frac{1 + 2 + \dots + n}{n} = \frac{\frac{n(n+1)}{2}}{n} = \frac{n+1}{2} .$$

Raising both sides to the nth power we get the desired result.

2.3. The simplest solution consists of using the weighted power means inequality to the (weighted) arithmetic and quadratic means of x and y with weights $\frac{p}{p+q}$ and $\frac{q}{p+q}$:

$$\frac{p}{p+q}\,x + \frac{q}{p+q}\,y \le \sqrt{\frac{p}{p+q}\,x^2 + \frac{q}{p+q}\,y^2} ,$$

hence

$$(px + qy)^2 \le (p + q)(px^2 + qy^2) .$$

Or we can use the Cauchy-Schwarz inequality as follows:

$$\begin{aligned}
(px + qy)^2 &= (\sqrt{p}\sqrt{p}\,x + \sqrt{q}\sqrt{q}\,y)^2 \\
&\le \left(\{\sqrt{p}\}^2 + \{\sqrt{q}\}^2\right)\left(\{\sqrt{p}\,x\}^2 + \{\sqrt{q}\,y\}^2\right) \quad \text{(Cauchy-Schwarz)} \\
&= (p + q)(px^2 + qy^2) .
\end{aligned}$$

Finally we use $p + q \le 1$ to obtain the desired result.

2.4. By the power means inequality:

$$\underbrace{\frac{a + b + c}{3}}_{M^1(a,b,c)} \quad \ge \quad \underbrace{\left(\frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{3}\right)^2}_{M^{1/2}(a,b,c)}$$

From here the desired result follows.

2.5. We have:

$$\begin{aligned}
x + y + z &= (x + y + z)\sqrt[3]{xyz} & (xyz = 1) \\
&\le \frac{(x + y + z)^2}{3} & \text{(AM-GM inequality)} \\
&\le x^2 + y^2 + z^2 . & \text{(power means inequality)}
\end{aligned}$$

2.6. The result can be obtained by using Minkowski's inequality repeatedly:

$$\begin{aligned}\sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+b_2^2}+\cdots+\sqrt{a_n^2+b_n^2} &\geq \sqrt{(a_1+a_2)^2+(b_1+b_2)^2}+\cdots+\sqrt{a_n^2+b_n^2}\\ &\geq \sqrt{(a_1+a_2+a_3)^2+(b_1+b_2+b_3)^2}+\cdots\\ &\qquad +\sqrt{a_n^2+b_n^2}\\ &\cdots\\ &\geq \sqrt{(a_1+a_2+\cdots+a_n)^2+(b_1+b_2+\cdots+b_n)^2}\end{aligned}$$

Another way to think about it is geometrically. Consider a sequence of points in the plane $P_k = (x_k, y_k)$, $k = 0, \dots, n$, such that

$$(x_k, y_k) = (x_{k-1} + a_k, y_{k-1} + b_k) \qquad \text{for } k = 1, \dots, n\,.$$

Then the left hand side of the inequality is the sum of the distances between two consecutive points, while the right hand side is the distance between the first one and the last one:

$$d(P_0, P_1) + d(P_1, P_2) + \cdots + d(P_{n-1}, P_n) \leq d(P_0, P_n)\,.$$

2.7. By the Arithmetic Mean-Geometric Mean Inequality

$$1 = \sqrt[n]{x_1 x_2 \dots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}\,,$$

Hence $f(x_1, x_2, \dots, x_n) \geq n$. On the other hand $f(1, 1, \dots, 1) = n$, so the minimum value is n.

2.8. For $x = y = z = 1$ we see that $S = 3/2$. We will prove that in fact $3/2$ is the minimum value of S by showing that $S \geq 3/2$.
Note that

$$S = \left(\frac{x}{\sqrt{y+z}}\right)^2 + \left(\frac{y}{\sqrt{z+x}}\right)^2 + \left(\frac{z}{\sqrt{x+y}}\right)^2\,.$$

Hence by the Cauchy-Schwarz inequality:

$$S \cdot (u^2 + v^2 + w^2) \geq \left(\frac{xu}{\sqrt{y+z}} + \frac{yv}{\sqrt{z+x}} + \frac{zw}{\sqrt{x+y}}\right)^2\,.$$

Writing $u = \sqrt{y+z}$, $v = \sqrt{z+x}$, $w = \sqrt{x+y}$ we get

$$S \cdot 2(x + y + z) \geq (x + y + z)^2\,,$$

hence, dividing by $2(x + y + z)$ and using the Arithmetic Mean-Geometric Mean inequality:

$$S \geq \frac{1}{2}(x + y + z) \geq \frac{1}{2} \cdot 3\sqrt[3]{xyz} = \frac{3}{2}\,.$$

2.9. By the Arithmetic Mean-Harmonic Mean inequality:

$$\frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} \leq \frac{x + y + z}{3} = \frac{1}{3}\,,$$

hence
$$9 \le \frac{1}{x} + \frac{1}{y} + \frac{1}{z} .$$
On the other hand for $x = y = z = 1/3$ the sum is 9, so the minimum value is 9.

2.10. Assume $a \le b \le c$, $A \le B \le C$. Then
$$\begin{aligned} 0 &\le (a-b)(A-B) + (a-c)(A-C) + (b-c)(B-C) \\ &= 3(aA + bB + cC) - (a+b+c)(A+B+C) . \end{aligned}$$
Using $A + B + C = \pi$ and dividing by $3(a + b + c)$ we get the desired result.
- *Remark:* We could have used also Chebyshev's Inequality:
$$\frac{a\,A + b\,B + c\,C}{3} \ge \left(\frac{a+b+c}{3}\right)\left(\frac{A+B+C}{3}\right) .$$

2.11. Assume $a_i + b_i > 0$ for each i (otherwise both sides are zero). Then by the Arithmetic Mean-Geometric Mean inequality
$$\left(\frac{a_1 \cdots a_n}{(a_1+b_1)\cdots(a_n+b_n)}\right)^{1/n} \le \frac{1}{n}\left(\frac{a_1}{a_1+b_1} + \cdots + \frac{a_n}{a_n+b_n} ,\right)$$
and similarly with the roles of a and b reversed. Adding both inequalities and clearing denominators we get the desired result.
(*Remark:* The result is known as *superadditivity* of the *geometric mean.*)

2.12. We have that $n!$ is increasing for $n \ge 1$, i.e., $1 \le n < m \implies n! < m!$ So $1999! > 2000 \implies (1999!)! > 2000! \implies ((1999!)!)! > (2000!)! \implies \dots \implies 1999!^{(2000)} > 2000!^{(1999)}$.

2.13. Consider the function $f(x) = (1999 - x)\ln(1999 + x)$. Its derivative is $f'(x) = -\ln(1999 + x) + \dfrac{1999 - x}{1999 + x}$, which is negative for $0 \le x \le 1$, because in that interval
$$\frac{1999 - x}{1999 + x} \le 1 = \ln e < \ln(1999 + x) .$$
Hence f is decreasing in $[0, 1]$ and $f(0) > f(1)$, i.e., $1999 \ln 1999 > 1998 \ln 2000$. Consequently $1999^{1999} > 2000^{1998}$.

2.14. Let $x = \log_2 3$ and $y = \log_3 5$, so $2^x = 3$, $3^y = 5$. Then, $27 = 3^3 = (2^x)^3 = 8^x$, and $25 = 5^2 = (3^y)^2 = 9^y$, hence $8^x > 9^y$, but $8 < 9$, hence $x > y$, i.e., $\log_2 3 > \log_3 5$.

2.15. We have $b^2 < \underbrace{b^2 + b + 1}_{a^2} < b^2 + 2b + 1 = (b+1)^2$. But b^2 and $(b+1)^2$ are consecutive squares, so there cannot be a square strictly between them.

2.16. We may assume that the leading coefficient is $+1$. The sum of the squares of the roots of $x^n + a_1x^{n-1} + \cdots + a_n$ is $a_1^2 - 2a_2$. The product of the squares of the roots is a_n^2. Using the Arithmetic Mean-Geometric Mean inequality we have

$$\frac{a_1^2 - 2a_2}{n} \geq \sqrt[n]{a_n^2}\,.$$

Since the coefficients are ± 1 that inequality is $(1 \pm 2)/n \geq 1$, hence $n \leq 3$.
Remark: $x^3 - x^2 - x + 1 = (x+1)(x-1)^2$ is an example of 3th degree polynomial with all coefficients equal to ± 1 and only real roots.

2.17. The given function is the square of the distance between a point of the quarter of circle $x^2+y^2 = 2$ in the open first quadrant and a point of the half hyperbola $xy = 9$ in that quadrant. The tangents to the curves at $(1,1)$ and $(3,3)$ separate the curves, and both are perpendicular to $x = y$, so those points are at the minimum distance, and the answer is $(3-1)^2 + (3-1)^2 = 8$.

2.18. Let

$$P = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{2n-1}{2n}\right), \quad Q = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\cdots\left(\frac{2n-2}{2n-1}\right).$$

We have $PQ = \dfrac{1}{2n}$. Also $\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \cdots < \frac{2n-1}{2n}$, hence $2P \geq Q$, so $2P^2 \geq PQ = \dfrac{1}{2n}$, and from here we get $P \geq \dfrac{1}{\sqrt{4n}}$.
On the other hand we have $P < Q\dfrac{2n}{2n+1} < Q$, hence $P^2 < PQ = \dfrac{1}{2n}$, and from here $P < \dfrac{1}{\sqrt{2n}}$.

2.19. The given inequality is equivalent to

$$\frac{(m+n)!}{m!\,n!}\,m^m n^n = \binom{m+n}{n} m^m n^n < (m+n)^{m+n}\,,$$

which is obviously true because the binomial expansion of $(m+n)^{m+n}$ includes the term on the left plus other terms.

2.20. Using the Arithmetic Mean-Geometric Mean inequality we get:

$$\frac{1}{n}\left\{\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}\right\} \geq \sqrt[n]{\frac{a_1}{b_1}\cdot\frac{a_2}{b_2}\cdots\frac{a_n}{b_n}} = 1\,.$$

From here the desired result follows.

2.21. We prove it by induction. For $n = 1$ the result is trivial, and for $n = 2$ it is a simple consequence of the following:

$$0 \leq (a_2 - a_1)(b_2 - b_1) = (a_1b_1 + a_2b_2) - (a_1b_2 + a_2b_1)\,.$$

Next assume that the result is true for some $n \geq 2$. We will prove that is is true for $n+1$. There are two possibilities:

1. If $x_{n+1} = b_{n+1}$, then we can apply the induction hypothesis to the n first terms of the sum and we are done.
2. If $x_{n+1} \neq b_{n+1}$, then $x_j = b_{n+1}$ for some $j \neq n+1$, and $x_{n+1} = b_k$ for some $k \neq n+1$. Hence:

$$\begin{aligned}\sum_{i=1}^{n+1} a_i x_i &= \sum_{\substack{i=1\\ i\neq j}}^{n} a_i x_i + a_j x_j + a_{n+1} x_{n+1} \\ &= \sum_{\substack{i=1\\ i\neq j}}^{n} a_i x_i + a_j b_{n+1} + a_{n+1} b_k\end{aligned}$$

(using the inequality for the two-term increasing sequences a_j, a_{n+1} and b_k, b_{n+1})

$$\leq \sum_{\substack{i=1\\ i\neq j}}^{n} a_i x_i + a_j b_k + a_{n+1} b_{n+1}\,.$$

This reduces the problem to case 1.

2.22. We have

$$\ln\left(\frac{1}{n}\sum_{k=1}^{n} a_k^p\right)^{1/p} = \frac{\ln\left(\frac{1}{n}\sum_{k=1}^{n} a_k^p\right)}{p}\,.$$

Also, $a_k \to 1$ as $p \to 0$, hence numerator and denominator tend to zero as p approaches zero. Using L'Hôpital we get

$$\lim_{p\to 0} \frac{\ln\left(\frac{1}{n}\sum_{k=1}^{n} a_k^p\right)}{p} = \lim_{p\to 0} \frac{\sum_{k=1}^{n} a_k^p \ln a_k}{\sum_{k=1}^{n} a_k^p} = \frac{\sum_{k=1}^{n} \ln a_k}{n} = \ln\left(\prod_{k=1}^{n} a_k\right)^{1/n}\,.$$

From here the desired result follows.

2.23. Set $x = b+c-a$, $y = c+a-b$, $z = a+b-c$. The triangle inequality implies that x, y, and z are positive. Furthermore, $a = (y+z)/2$, $b = (z+x)/2$, and $c = (x+y)/2$. The LHS of the inequality becomes:

$$\frac{y+z}{2x} + \frac{z+x}{2y} + \frac{x+y}{2z} = \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{x}{z} + \frac{z}{x}\right) \geq 3\,.$$

2.24. We have that $2 \uparrow\uparrow 3 = 2^{2^2} = 16 < 27 = 3^3 = 3 \uparrow\uparrow 2$. Then using $a \uparrow\uparrow (n+1) = a^{a\uparrow\uparrow n}$ we get $2 \uparrow\uparrow (n+1) < 3 \uparrow\uparrow n$ for $n \geq 2$, and from here it follows that $2 \uparrow\uparrow 2011 < 3 \uparrow\uparrow 2010$.

2.25. Consider the function $f(x) = e^{1/x}$ for $x > 0$. We have $f'(x) = -\frac{1/x}{e^{1/x}} < 0$, $f''(x) = e^{1/x}(\frac{2}{x^3} + \frac{1}{x^4}) > 0$, hence f is decreasing and convex.

By convexity, we have

$$\frac{1}{2}(f(e)+f(\pi)) \geq f\left(\tfrac{e+\pi}{2}\right) .$$

On the other hand we have $(e+\pi)/2 < 3$, and since f is decreasing, $f(\frac{e+\pi}{2}) > f(3)$, and from here the result follows.

2.26. We have:

$$\begin{aligned} f(x)-f(\overline{a}) &= \sum_{i=1}^{n}(x-a_i)^2 - \sum_{i=1}^{n}(\overline{a}-a_i)^2 \\ &= \sum_{i=1}^{n}\left\{(x-a_i)^2-(\overline{a}-a_i)^2\right\} \\ &= \sum_{i=1}^{n}(x^2-2a_ix-\overline{a}^2+2a_i\overline{a}) \\ &= nx^2-2n\overline{a}x+n\overline{a}^2 \\ &= n(x-\overline{a})^2 \geq 0\,, \end{aligned}$$

hence $f(x) \geq f(\overline{a})$ for every x.

2.27. By the Geometric Mean-Arithmetic Mean inequality

$$x_1+\frac{1}{x_2} \geq 2\sqrt{\frac{x_1}{x_2}}\,,\dots,x_{100}+\frac{1}{x_1} \geq 2\sqrt{\frac{x_{100}}{x_1}}\,.$$

Multiplying we get

$$\left(x_1+\frac{1}{x_2}\right)\left(x_2+\frac{1}{x_3}\right)\cdots\left(x_{100}+\frac{1}{x_1}\right) \geq 2^{100}\,.$$

From the system of equations we get

$$\left(x_1+\frac{1}{x_2}\right)\left(x_2+\frac{1}{x_3}\right)\cdots\left(x_{100}+\frac{1}{x_1}\right) = 2^{100}\,,$$

so all those inequalities are equalities, i.e.,

$$x_1+\frac{1}{x_2} = 2\sqrt{\frac{x_1}{x_2}} \quad\Longrightarrow\quad \left(\sqrt{x_1}-\frac{1}{\sqrt{x_2}}\right)^2 = 0 \quad\Longrightarrow\quad x_1 = \frac{1}{x_2}\,,$$

and analogously: $x_2 = 1/x_3$, $\dots$, $x_{100} = 1/x_1$. Hence $x_1 = 1/x_2$, $x_2 = 1/x_3$, $\dots$, $x_{100} = 1/x_1$, and from here we get $x_1 = 2$, $x_2 = 1/2$,$\dots$, $x_{99} = 2$, $x_{100} = 1/2$.

2.28. Squaring the inequalities and moving their left hand sides to the right we get

$$\begin{aligned} 0 &\geq c^2-(a-b)^2 = (c+a-b)(c-a+b) \\ 0 &\geq a^2-(b-c)^2 = (a+b-c)(a-b+c) \\ 0 &\geq b^2-(c-a)^2 = (b+c-a)(b-c+a)\,. \end{aligned}$$

Multiplying them together we get:

$$0 \geq (a+b-c)^2(a-b+c)^2(-a+b+c)^2\,,$$

hence, one of the factors must be zero.

2.29. The answer is 1. In fact, the sequences $(\sin^3 x, \cos^3 x)$ and $(1/\sin x, 1/\cos x)$ are oppositely sorted, hence by the rearrangement inequality:

$$\begin{aligned}\sin^3 x/\cos x + \cos^3 x/\sin x &\geq \sin^3 x/\sin x + \cos^3 x/\cos x \\ &= \sin^2 x + \cos^2 x = 1\,.\end{aligned}$$

Equality is attained at $x = \pi/4$.

2.30. By the rearrangement inequality we have for $k = 2, 3, \cdots, n$:

$$\frac{a_1}{s-a_1} + \frac{a_2}{s-a_2} + \cdots + \frac{a_n}{s-a_n} \geq \frac{a_1}{s-a_k} + \frac{a_2}{s-a_{k+1}} + \cdots + \frac{a_n}{s-a_{k-1}}\,,$$

were the numerators on the right hand side are a cyclic permutation of $a_1, \cdots, a_n$ (assume $a_{n+i} = a_i$). Adding those $n-1$ inequalities we get

$$(n-1)\left(\frac{a_1}{s-a_1} + \frac{a_2}{s-a_2} + \cdots + \frac{a_n}{s-a_n}\right) \geq \frac{s-a_1}{s-a_1} + \frac{s-a_2}{s-a_2} + \cdots + \frac{s-a_n}{s-a_n} = n\,,$$

and the result follows.

2.31. The answer is 1. Since $|\sin x| \leq 1$ we have $\sin^4(x) \leq \sin^2(x)$, and analogously $\cos^4(x) \leq \cos^2(x)$. Hence $f(x) = \sin^4(x) + \cos^4 x \leq \sin^2(x) + \cos^2 x = 1$. On the other hand the value 1 s attained e.g. at $x = 0$.

2.32. Let $u = v = \sqrt{ab}$, $w = c$. By the AGM inequality we have

$$\sqrt[3]{uvw} \leq \frac{u+v+w}{3} \Rightarrow \sqrt[3]{abc} \leq \frac{2\sqrt{ab}+c}{3} \Rightarrow 3\sqrt[3]{abc} - (a+b+c) \leq 2\sqrt{ab} - (a+b)\,.$$

The last inequality is equivalent to the desired result.
Equality happens precisely for $u = v = w$, i.e., $c = \sqrt{ab}$.

2.33. The AGM inequality applied to $a_1, \dots, a_n$ shows that $b_n \geq 0$. Also, letting $u_k = \sqrt[n]{a_1 \cdots a_n}$, $k = 1, \dots, n$, $u_{n+1} = a_{n+1}$ and using again the AGM inequality we get

$$\begin{aligned}\sqrt[n+1]{u_1 \cdots u_n u_{n+1}} &\leq \frac{u_1 + \cdots + u_n + u_{n+1}}{n+1} \\ \sqrt[n+1]{a_1 \cdots a_n a_{n+1}} &\leq \frac{n\sqrt[n]{a_1 \cdots a_n} + a_{n+1}}{n+1}\,.\end{aligned}$$

Multiplying both sides by $n+1$ and subtracting $a_1 + \cdots + a_n + a_{n+1}$ we get $-b_{n+1} \leq -b_n$, which is equivalent to the desired result.

3.1. If p and q are consecutive primes and $p + q = 2r$, then $r = (p+q)/2$ and $p < r < q$, but there are no primes between p and q.

3.2. (a) No, a square divisible by 3 is also divisible by 9.
(b) Same argument.

3.3. Assume that the set of primes of the form $4n+3$ is finite. Let P be their product. Consider the number $N = P^2 - 2$. Note that the square of an odd number is of the form $4n+1$, hence P^2 is of the form $4n+1$ and N will be of the form $4n+3$. Now, if all prime factors of N where of the form $4n+1$, N would be of the form $4n+1$, so N must have some prime factor p of the form $4n+3$. So it must be one of the primes in the product P, hence p divides $N - P^2 = 2$, which is impossible.

3.4. That is equivalent to proving that $n^3 + 2n$ and $n^4 + 3n^2 + 1$ are relatively prime for every n. These are two possible ways to show it:
- Assume a prime p divides $n^3 + 2n = n(n^2 + 2)$. Then it must divide n or $n^2 + 2$. Writing $n^4 + 3n^2 + 1 = n^2(n^2 + 3) + 1 = (n^2 + 1)(n^2 + 2) - 1$ we see that p cannot divide $n^4 + 3n^2 + 1$ in either case.
- The following identity

$$(n^2 + 1)(n^4 + 3n^2 + 1) - (n^3 + 2n)^2 = 1$$

(which can be checked algebraically) shows that any common factor of $n^4 + 3n^2 + 1$ and $n^3 + 2n$ should divide 1, so their gcd is always 1. (Note: if you are wondering how I arrived to that identity, I just used the Euclidean algorithm on the two given polynomials.)

3.5. Assume $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, with $a_n \neq 0$. We will assume WLOG that $a_n > 0$, so that $p(k) > 0$ for every k large enough—otherwise we can use the argument below with $-p(x)$ instead of $p(x)$.
We have

$$p(p(k) + k) = \sum_{i=0}^{n} a_i \left[p(k) + k\right]^i .$$

For each term of that sum we have that

$$a_i \left[p(k) + k\right]^i = [\text{multiple of } p(k)] + a_i k^i ,$$

and the sum of the $a_i k^i$ is precisely $p(k)$, so $p(p(k) + k)$ is a multiple of $p(k)$. It remains only to note that $p(p(k) + k) \neq p(k)$ for infinitely many positive integers k, otherwise $p(p(x) + x)$ and $p(x)$ would be the same polynomial, which is easily ruled out for non constant $p(x)$.

3.6. This can be proved easily by induction. Base case: $F_1 = 1$ and $F_2 = 1$ are in fact relatively prime. Induction Step: we must prove that if F_n and F_{n+1} are relatively prime then so are F_{n+1} and F_{n+2}. But this follows from the recursive definition of the Fibonacci sequence: $F_n + F_{n+1} = F_{n+2}$; any common factor of F_{n+1} and F_{n+2} would be also a factor of F_n, and consequently it would be a common factor of F_n and F_{n+1} (which by induction hypothesis are relatively prime.)

3.7. For any integer n we have that n^2 only can be 0 or 1 mod 3. So if 3 does not divide a or b they must be 1 mod 3, and their sum will be 2 modulo 3, which cannot be a square.

3.8. Assume the sum S is an integer. Let 2^i be the maximum power of 2 dividing n, and let 2^j be the maximum power of 2 dividing n! Then

$$\frac{n!}{2^j}2^i S = \sum_{k=1}^{n} \frac{n!}{k2^{j-i}} .$$

For $n \geq 2$ the left hand side is an even number. In the right hand side all the terms of the sum are even integers except the one for $k = 2^i$ which is an odd integer, so the sum must be odd. Hence we have an even number equal to an odd number, which is impossible.

3.9. Since each digit cannot be greater than 9, we have that $f(n) \leq 9 \cdot (1 + \log_{10} n)$, so in particular $f(N) \leq 9 \cdot (1 + 4444 \cdot \log_{10} 4444) < 9 \cdot (1 + 4444 \cdot 4) = 159993$. Next we have $f(f(N)) \leq 9 \cdot 6 = 54$. Finally among numbers not greater than 54, the one with the greatest sum of the digits is 49, hence $f(f(f(N))) \leq 4 + 9 = 13$.
Next we use that $n \equiv f(n) \pmod 9$. Since $4444 \equiv 7 \pmod 9$, then

$$4444^{4444} \equiv 7^{4444} \pmod 9 .$$

We notice that the sequence $7^n \bmod 9$ for $n = 0, 1, 2, \dots$ is $1, 7, 4, 1, 7, 4, \dots$, with period 3. Since $4444 \equiv 1 \pmod 3$, we have $7^{4444} \equiv 7^1 \pmod 9$, hence $f(f(f(N))) \equiv 7 \pmod 9$. The only positive integer not greater than 13 that is congruent with 7 modulo 9 is 7, hence $f(f(f(N))) = 7$.

3.10. Pick 1999 different prime numbers $p_1, p_2, \dots, p_{1999}$ (we can do that because the set of prime numbers is infinite) and solve the following system of 1999 congruences:

$$\begin{cases} x & \equiv & 0 & \pmod{p_1^3} \\ x & \equiv & -1 & \pmod{p_2^3} \\ x & \equiv & -2 & \pmod{p_3^3} \\ & \dots & & \\ x & \equiv & -1998 & \pmod{p_{1999}^3} \end{cases}$$

According to the Chinese Remainder Theorem, that system of congruences has a solution x (modulo $M = p_1^3 \dots p_{1999}^3$). For $k = 1, \dots, 1999$ we have that $x + k \equiv 0 \pmod{p_k^3}$, hence $x + k$ is in fact a multiple of p_k^3.

3.11. Assume $a \geq b \geq c$. Then

$$2 = \left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) \leq \left(1 + \frac{1}{c}\right)^3 .$$

From here we get that $c < 4$, so its only possible values are $c = 1, 2, 3$.

For $c = 1$ we get $(1 + 1/c) = 2$, hence

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right) = 1\,,$$

which is impossible.
For $c = 2$ we have $(1 + 1/c) = 3/2$, hence

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right) = \frac{4}{3}\,,$$

and from here we get

$$a = \frac{3(b+1)}{b-3}\,,$$

with solutions $(a, b) = (15, 4)$, $(9, 5)$ and $(7, 6)$.
Finally for $c = 3$ we have $1 + 1/c = 1 + 1/3 = 4/3$, hence

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right) = \frac{3}{2}\,.$$

So

$$a = \frac{2(b+1)}{b-2}\,.$$

The solutions are $(a, b) = (8, 3)$ and $(5, 4)$.
So the complete set of solutions verifying $a \geq b \geq c$ are

$$(a, b, c) = (15, 4, 2),\ (9, 5, 2),\ (7, 6, 2),\ (8, 3, 3),\ (5, 4, 3)\,.$$

The rest of the triples verifying the given equation can be obtained by permutations of a, b, c.

3.12. The change of variables $x = a + 1$, $y = b + 1$, $z = c + 1$, transforms the equation into the following one:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1\,.$$

Assuming $x \leq y \leq z$ we have that $x \leq 3 \leq z$.
For $x = 1$ the equation becomes

$$\frac{1}{y} + \frac{1}{z} = 0\,.$$

which is impossible.
For $x = 2$ we have

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{2}\,,$$

or

$$z = \frac{2y}{y-2}\,,$$

with solutions $(y, z) = (3, 6)$ and $(4, 4)$.
For $x = 3$ the only possibility is $(y, z) = (3, 3)$.

So the list of solutions is

$$(x,y,z) = (2,3,6),\ (2,4,4),\ (3,3,3)\,,$$

and the ones obtained by permuting x, y, z.
With the original variables the solutions are (except for permutations of variables);

$$(a,b,c) = (1,2,5),\ (1,3,3),\ (2,2,2)\,.$$

3.13. We look at the equation modulo 16. First we notice that $n^4 \equiv 0$ or $1 \pmod{16}$ depending on whether n is even or odd. On the other hand $1599 \equiv 15 \pmod{16}$. So the equation can be satisfied only if the number of odd terms in the LHS is 15 modulo 16, but that is impossible because there are only 14 terms in the LHS. Hence the equation has no solution.

3.14. Call $N = 10^{10^{10^{10}}}$, and consider the sequence $a_n =$ remainder of dividing F_n by N. Since there are only N^2 pairs of non-negative integers less than N, there must be two identical pairs $(a_i, a_{i+1}) = (a_j, a_{j+1})$ for some $0 \le i < j$. Let $k = j - i$. Since $a_{n+2} = a_{n+1} + a_n$ and $a_{n-1} = a_{n+1} - a_n$, by induction we get that $a_n = a_{n+k}$ for every $n \ge 0$, so in particular $a_k = a_0 = 0$, and this implies that F_k is a multiple of N. (In fact since there are $N^2 + 1$ pairs (a_i, a_{i+1}), for $i = 0, 1, \dots, N^2$, we can add the restriction $0 \le i < j \le N$ above and get that the result is true for some k such that $0 < k \le N^2$.)

3.15. The answer is affirmative. Let $a = \sqrt{6}$ and $b = \sqrt{3}$. Assume $\lfloor a^m \rfloor = \lfloor b^n \rfloor = k$ for some positive integers m, n. Then, $k^2 \le 6^m < (k+1)^2 = k^2 + 2k + 1$, and $k^2 \le 3^n < (k+1)^2 = k^2 + 2k + 1$. Hence, subtracting the inequalities and taking into account that $n > m$:

$$2k \ge |6^m - 3^n| = 3^m |2^m - 3^{n-m}| \ge 3^m\,.$$

Hence $\dfrac{9^m}{4} \le k^2 \le 6^m$, which implies $\frac{1}{4} \le \left(\frac{2}{3}\right)^m$. This holds only for $m = 1, 2, 3$. This values of m can be ruled out by checking the values of

$$\lfloor a \rfloor = 2, \quad \lfloor a^2 \rfloor = 6, \quad \lfloor a^3 \rfloor = 14,$$

$$\lfloor b \rfloor = 1, \quad \lfloor b^2 \rfloor = 3, \quad \lfloor b^3 \rfloor = 5, \quad \lfloor b^4 \rfloor = 9, \quad \lfloor b^5 \rfloor = 15\,.$$

Hence, $\lfloor a^m \rfloor \ne \lfloor b^n \rfloor$ for every positive integers m, n.

3.16. There are integers k, r such that $10^k < 2^{2005} < 10^{k+1}$ and $10^r < 5^{2005} < 10^{r+1}$. Hence $10^{k+r} < 10^{2005} < 10^{k+r+2}$, $k + r + 1 = 2005$. Now the number of digits in 2^{2005} is $k + 1$, and the number of digits in 5^{2005} is $r + 1$. Hence the total number of digits is 2^{2005} and 5^{2005} is $k + r + 2 = 2006$.

3.17. For $p = 2$, $p^2 + 2 = 6$ is not prime.
For $p = 3$, $p^2 + 2 = 11$, and $p^3 + 2 = 29$ are all prime and the statement is true.
For prime $p > 3$ we have that p is an odd number not divisible by 3, so it is congruent to ± 1 modulo 6. Hence $p^2 + 2 \equiv 3 \pmod{6}$ is multiple of 3 and cannot be prime.

3.18. If n is even then $n^4 + 4^n$ is even and greater than 2, so it cannot be prime.
If n is odd, then $n = 2k+1$ for some integer k, hence $n^4 + 4^n = n^4 + 4 \cdot (2^k)^4$. Next, use Sophie Germain's identity: $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$.

3.19. From the hypothesis we have that $m + 1 \leq \lfloor \sqrt{n} + \frac{1}{2} \rfloor \leq \sqrt{n} + \frac{1}{2}$. But the second inequality must be strict because $\sqrt{n}$ is irrational or an integer, and consequently $\sqrt{n} + \frac{1}{2}$ cannot be an integer. From here the desired result follows.

3.20. Assume $m \neq \lfloor n + \sqrt{n} + 1/2 \rfloor$ for every $n = 1, 2, 3, \dots$ Then for some n, $f(n) < m < f(n+1)$. The first inequality implies

$$n + \sqrt{n} + \frac{1}{2} < m\,.$$

The second inequality implies $m + 1 \leq f(n+1)$, and

$$m + 1 < n + 1 + \sqrt{n+1} + \frac{1}{2}$$

(Note that equality is impossible because the right hand side cannot be an integer.) Hence

$$\sqrt{n} < m - n - \frac{1}{2} < \sqrt{n+1}\,,$$

$$n < (m-n)^2 - (m-n) + \frac{1}{4} < n + 1$$

$$n - \frac{1}{4} < (m-n)^2 - (m-n) < n + \frac{3}{4}$$

$$(m-n)^2 - (m-n) = n\,.$$

$$m = (m-n)^2\,.$$

So, m is a square.
We are not done yet, since we still must prove that $f(n)$ misses *all* the squares. To do so we use a counting argument. Among all positive integers $\leq k^2 + k$ there are exactly k squares, and exactly k^2 integers of the form $f(n) = \lfloor n + \sqrt{n} + 1/2 \rfloor$. Hence $f(n)$ is the nth non square.
Another way to express it: in the set $A(k) = \{1, 2, 3, \dots, k^2 + k\}$ consider the two subsets $S(k)$ = squares in $A(k)$, and $N(k)$ = integers of the form $f(n) = \lfloor n + \sqrt{n} + 1/2 \rfloor$ in $A(k)$. The set $S(k)$ has k elements, $N(k)$ has k^2 elements, and $A(k) = S(k) \cup N(k)$. Since

$$\underbrace{|S(k) \cup N(k)|}_{k^2+k} = \underbrace{|S(k)|}_{k} + \underbrace{|N(k)|}_{k^2} - |S(k) \cap N(k)|$$

we get that $|S(k) \cap N(k)| = 0$, i.e, $S(k) \cap N(k)$ must be empty.

3.21. Each of the given numbers can be written

$$1 + 1000 + 1000^2 + \cdots + 1000^n = p_n(10^3)$$

where $p_n(x) = 1+x+x^2+\cdots+x^n$, $n = 1, 2, 3, \dots$. We have $(x-1)p_n(x) = x^{n+1}-1$. If we set $x = 10^3$, we get:

$$999 \cdot p_n(10^3) = 10^{3(n+1)} - 1 = (10^{n+1} - 1)(10^{2(n+1)} + 10^{n+1} + 1)\,.$$

If $p_n(10^3)$ were prime it should divide one of the factors on the RHS. It cannot divide $10^{n+1}-1$, because this factor is less than $p_n(10^3)$, so $p_n(10^3)$ must divide the other factor. Hence $10^{n+1} - 1$ must divide 999, but this is impossible for $n > 2$. In only remains to check the cases $n = 1$ and $n = 2$. But $1001 = 7 \cdot 11 \cdot 13$, and $1001001 = 3 \cdot 333667$, so they are not prime either.

3.22. We will prove that the sequence is eventually constant if and only if b_0 is a perfect square.
The "if" part is trivial, because if b_k is a perfect square then $d(b_k) = 0$, and $b_{k+1} = b_k$.
For the "only if" part assume that b_k is *not* a perfect square. Then, suppose that $r^2 < b_k < (r+1)^2$. Then, $d(b_k) = b_k - r^2$ is in the interval $[1, 2r]$, so $b_{k+1} = r^2+2d(b_k)$ is greater than r^2 but less than $(r + 2)^2$, and not equal to $(r + 1)^2$ by parity. Thus b_{k+1} is also not a perfect square, and is greater than b_k. So, if b_0 is not a perfect square, no b_k is a perfect square and the sequence diverges to infinity.

3.23. The answer is 41. In fact, we have:

$$\gcd(a_n, a_{n+1}) = \gcd(a_n, a_{n+1} - a_n) = \gcd(n^2 + 10, 2n + 1) = \cdots$$

(since $2n + 1$ is odd we can multiply the other argument by 4 without altering the gcd)

$$\begin{aligned}\cdots = \gcd(4n^2 + 40, 2n + 1) &= \gcd((2n + 1)(2n - 1) + 41, 2n + 1)\\ &= \gcd(41, 2n + 1) \leq 41\,.\end{aligned}$$

The maximum value is attained e.g. at $n = 20$.

3.24. The given condition implies:

$$(x - y)(2x + 2y + 1) = y^2\,.$$

Since the right hand side is a square, to prove that the two factors on the left hand side are also squares it suffices to prove that they are relatively prime. In fact, if p if a prime number dividing $x - y$ then it divides y^2 and consequently it divides y. So p also divides x, and $x + y$. But then it cannot divide $2x + 2y + 1$.
An analogous reasoning works using the following relation, also implied the given condition:

$$(x - y)(3x + 3y + 1) = x^2\,.$$

3.25. It suffices to prove that n is a multiple of 5 and 8, in other words, that $n \equiv 0 \pmod 5$, and $n \equiv 0 \pmod 8$.

We first think modulo 5. Perfect squares can be congruent to 0, 1, or 4 modulo 5 only. We have:

$$\begin{aligned}
2n+1 &\equiv 0 \pmod 5 &\implies& \quad n \equiv 2 \pmod 5 \\
2n+1 &\equiv 1 \pmod 5 &\implies& \quad n \equiv 0 \pmod 5 \\
2n+1 &\equiv 4 \pmod 5 &\implies& \quad n \equiv 4 \pmod 5 \\
3n+1 &\equiv 0 \pmod 5 &\implies& \quad n \equiv 3 \pmod 5 \\
3n+1 &\equiv 1 \pmod 5 &\implies& \quad n \equiv 0 \pmod 5 \\
3n+1 &\equiv 4 \pmod 5 &\implies& \quad n \equiv 1 \pmod 5 .
\end{aligned}$$

So the only possibility that can make both $2n+1$ and $3n+1$ perfect squares is $n \equiv 0 \pmod 5$, i.e., n is a multiple of 5.
Next, we think modulo 8. Perfect squares can only be congruent to 0, 1, or 4 modulo 8, and we have:

$$\begin{aligned}
3n+1 &\equiv 0 \pmod 8 &\implies& \quad n \equiv 5 \pmod 8 \\
3n+1 &\equiv 1 \pmod 8 &\implies& \quad n \equiv 0 \pmod 8 \\
3n+1 &\equiv 4 \pmod 8 &\implies& \quad n \equiv 1 \pmod 8 .
\end{aligned}$$

The possibilities $n \equiv 5 \pmod 8$ and $n \equiv 1 \pmod 8$ can be ruled out because n must be even. In fact, if $2n+1 = a^2$, then a is odd, and $2n = a^2 - 1 = (a+1)(a-1)$. Since a is odd we have that $a-1$ and $a+1$ are even, so $2n$ must be a multiple of 4, consequently n is even. So, we have that the only possibility is $n \equiv 0 \pmod 8$, i.e., n is a multiple of 8.
Since n is a multiple of 5 and 8, it must be indeed a multiple of 40, QED.

3.26. The prime factorization of 1000! contains more 2's than 5's, so the number of zeros at the end of 1000! will equal the exponent of 5. That will be equal to the number of multiples of 5 in the sequence $1, 2, 3, \dots, 1000$, plus the number of multiples of $5^2 = 25$, plus the number of multiples of $5^3 = 125$, and the multiples of $5^4 = 625$, in total:

$$\left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{25} \right\rfloor + \left\lfloor \frac{1000}{125} \right\rfloor + \left\lfloor \frac{1000}{625} \right\rfloor = 200 + 40 + 8 + 1 = 249 .$$

So 1000! ends with 249 zeros.

3.27. The answer is 8.
More generally, for any given positive integer n, the number of binomial coefficients $\binom{n}{k}$ that are odd equals 2 raised to the number of 1's in the binary representation of n—so, for $n = 100$, with binary representation 1100100 (three 1's), the answer is $2^3 = 8$. We prove it by induction in the number s of 1's in the binary representation of n.

- Basis step: If $s = 1$, then n is a power of 2, say $n = 2^r$. Next, we use that the exponent of a prime number p in the prime factorization of $m!$ is

$$\sum_{i \geq 1} \left\lfloor \frac{m}{p^i} \right\rfloor ,$$

where $\lfloor x \rfloor$ = greatest integer $\leq x$. Since $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, the exponent of 2 in the prime factorization of $\binom{2^r}{k}$ is

$$\sum_{i \geq 1} \left\lfloor \frac{2^r}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{k}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{2^r - k}{2^i} \right\rfloor = \sum_{i=1}^{r} \left\lceil \frac{k}{2^i} \right\rceil - \sum_{i=1}^{r} \left\lfloor \frac{k}{2^i} \right\rfloor ,$$

where $\lceil x \rceil$ = least integer $\geq x$. The right hand side is the number of values of i in the interval from 1 to r for which $\frac{k}{2^i}$ is not an integer. If $k = 0$ or $k = 2^r$ then the expression is 0, i.e., $\binom{2^r}{k}$ is odd. Otherwise, for $0 < k < 2^r$, the right hand side is strictly positive (at least $k/2^r$ is not an integer), and in that case $\binom{2^r}{k}$ is even. So the number of values of k for which $\binom{2^r}{k}$ is odd is $2 = 2^1$. This sets the basis step of the induction process.

- Induction step: Assume the statement is true for a given $s \geq 1$, and assume that the number of 1's in the binary representation of n is $s + 1$, so n can be written $n = 2^r + n'$, where $0 < n' < 2^r$ and n' has s 1's in its binary representation. By induction hypothesis the number of values of k for which $\binom{n'}{k}$ is odd is 2^s. We must prove that the number of values of k for which $\binom{n}{k} = \binom{2^r+n'}{k}$ is odd is 2^{s+1}. To do so we will study the parity of $\binom{n}{k}$ in three intervals, namely $0 \leq k \leq n'$, $n' < k < 2^r$, and $2^r \leq k \leq n$.

(1) For every k such that $0 \leq k \leq n'$, $\binom{n'}{k}$ and $\binom{n}{k}$ have the same parity. In fact, using again the above formula to determine the exponent of 2 in the prime factorization of binomial coefficients, we get

$$\begin{aligned}
\sum_{i \geq 1} \left\lfloor \frac{n}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{k}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{n-k}{2^i} \right\rfloor &= \sum_{i \geq 1} \left\lfloor \frac{2^r + n'}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{k}{2^i} \right\rfloor \\
&\quad - \sum_{i \geq 1} \left\lfloor \frac{2^r + n' - k}{2^i} \right\rfloor \\
&= \sum_{i=1}^{r} \left(2^{r-i} + \left\lfloor \frac{n'}{2^i} \right\rfloor \right) - \sum_{i=1}^{r} \left\lfloor \frac{k}{2^i} \right\rfloor \\
&\quad - \sum_{i=1}^{r} \left(2^{r-i} + \left\lfloor \frac{n'-k}{2^i} \right\rfloor \right) \\
&= \sum_{i \geq 1} \left\lfloor \frac{n'}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{k}{2^i} \right\rfloor - \sum_{i \geq 1} \left\lfloor \frac{n'-k}{2^i} \right\rfloor .
\end{aligned}$$

Hence, the number of values of k in the interval from 0 to n' for which $\binom{n}{k}$ is odd is 2^s.

(2) If $n' < k < 2^r$, then $\binom{n}{k}$ is even. In fact, we have that the power of 2 in the prime factorization of $\binom{n}{k}$ is:

$$\sum_{i\geq 1} \left\lfloor \frac{n}{2^i} \right\rfloor - \sum_{i\geq 1} \left\lfloor \frac{k}{2^i} \right\rfloor - \sum_{i\geq 1} \left\lfloor \frac{n-k}{2^i} \right\rfloor = \sum_{i=1}^{r} \left(\left\lfloor \frac{n}{2^i} \right\rfloor - \left\lfloor \frac{k}{2^i} \right\rfloor - \left\lfloor \frac{n-k}{2^i} \right\rfloor \right) .$$

using $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x+y \rfloor$, and given that $n/2^i = k/2^i + (n-k)/2^i$, we see that all terms of the sum on the right hand side are nonnegative, and all we have to show is that at least one of them is strictly positive. That can be accomplished by taking $i = r$. In fact, we have $2^r < n < 2^{r+1}$, hence $1 < n/2^r < 2$, $\lfloor n/2^r \rfloor = 1$. Also, $0 < k < 2^r$, hence $0 < k/2^r < 1$, $\lfloor k/2^r \rfloor = 0$. And $2^r = n - n' > n - k > 0$, so $0 < (n-k)/2^r < 1$, $\lfloor (n-k)/2^r \rfloor = 0$. Hence,

$$\left\lfloor \frac{n}{2^r} \right\rfloor - \left\lfloor \frac{k}{2^r} \right\rfloor - \left\lfloor \frac{n-k}{2^r} \right\rfloor = 1 - 0 - 0 = 1 > 0 .$$

(3) If $2^r \leq k \leq n$, then letting $k' = n - k$ we have that $0 \leq k' \leq n'$, and $\binom{n}{k} = \binom{n}{k'}$, and by (1), the number of values of k in the interval from 2^r to n for which $\binom{n}{k}$ is odd is 2^s.

The three results (1), (2) and (3) combined show that the number of values of k for which $\binom{n}{k}$ is odd is $2 \cdot 2^s = 2^{s+1}$. This completes the induction step, and the result is proved.

3.28. The answer is 3.

Note that $2^5 = 32$, $5^5 = 3125$, so 3 is in fact a solution. We will prove that it is the only solution.

Let d be the common digit at the beginning of 2^n and 5^n. Then

$$d \cdot 10^r \leq 2^n < (d+1) \cdot 10^r ,$$
$$d \cdot 10^s \leq 5^n < (d+1) \cdot 10^s$$

for some integers r, s. Multiplying the inequalities we get

$$d^2 10^{r+s} \leq 10^n < (d+1)^2 10^{r+s} ,$$

$$d^2 \leq 10^{n-r-s} < (d+1)^2 ,$$

so d is such that between d^2 and $(d+1)^2$ there must be a power of 10. The only possible solutions are $d = 1$ and $d = 3$. The case $d = 1$ can be ruled out because that would imply $n = r + s$, and from the inequalities above would get

$$5^r \leq 2^s < 2 \cdot 5^r ,$$
$$2^s \leq 5^r < 2 \cdot 2^s ,$$

hence $2^s = 5^r$, which is impossible unless $r = s = 0$ (implying $n = 0$).

Hence, the only possibility is $d = 3$.

3.29. Assume that the given binomial coefficients are in arithmetic progression. Multiplying each binomial number by $(k+3)!(n-k)!$ and simplifying we get that the following numbers are also in arithmetic progression:

$$\begin{aligned}&(k+1)(k+2)(k+3),\\&(n-k)(k+2)(k+3),\\&(n-k)(n-k-1)(k+3),\\&(n-k)(n-k-1)(n-k-2)\,.\end{aligned}$$

Their differences are

$$\begin{aligned}&(n-2k-1)(k+2)(k+3),\\&(n-k)(n-2k-3)(k+3),\\&(n-k)(n-k-1)(n-2k-5)\,.\end{aligned}$$

Writing that they must be equal we get a system of two equations:

$$\begin{cases}n^2-4kn-5n+4k^2+8k+2=0\\n^2-4kn-9n+4k^2+16k+14=0\end{cases}$$

Subtracting both equations we get

$$4n-8k-12=0\,,$$

i.e., $n=2k+3$, so the four binomial numbers should be of the form

$$\binom{2k+3}{k},\quad\binom{2k+3}{k+1},\quad\binom{2k+3}{k+2},\quad\binom{2k+3}{k+3}.$$

However

$$\binom{2k+3}{k}<\binom{2k+3}{k+1}=\binom{2k+3}{k+2}>\binom{2k+3}{k+3},$$

so they cannot be in arithmetic progression.
- *Remark:* There are sets of three consecutive binomial numbers in arithmetic progression, e.g.: $\binom{7}{1}=7$, $\binom{7}{2}=21$, $\binom{7}{3}=35$.

3.30. We proceed by induction, with base case $1=2^03^0$. Suppose all integers less than $n-1$ can be represented. If n is even, then we can take a representation of $n/2$ and multiply each term by 2 to obtain a representation of n. If n is odd, put $m=\lfloor\log_3 n\rfloor$, so that $3^m\le n<3^{m+1}$. If $3^m=n$, we are done. Otherwise, choose a representation $(n-3^m)/2=s_1+\cdots+s_k$ in the desired form. Then

$$n=3^m+2s_1+\cdots+2s_k,$$

and clearly none of the $2s_i$ divide each other or 3^m. Moreover, since $2s_i\le n-3^m<3^{m+1}-3^m$, we have $s_i<3^m$, so 3^m cannot divide $2s_i$ either. Thus n has a representation of the desired form in all cases, completing the induction.

3.31. There are n such sums. More precisely, there is exactly one such sum with k terms for each of $k=1,\dots,n$ (and clearly no others). To see this, note that if

$n = a_1 + a_2 + \cdots + a_k$ with $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$, then

$$\begin{aligned} ka_1 &= a_1 + a_1 + \cdots + a_1 \\ &\le n \le a_1 + (a_1 + 1) + \cdots + (a_1 + 1) \\ &= ka_1 + k - 1. \end{aligned}$$

However, there is a unique integer a_1 satisfying these inequalities, namely $a_1 = \lfloor n/k \rfloor$. Moreover, once a_1 is fixed, there are k different possibilities for the sum $a_1 + a_2 + \cdots + a_k$: if i is the last integer such that $a_i = a_1$, then the sum equals $ka_1 + (i - 1)$. The possible values of i are $1, \dots, k$, and exactly one of these sums comes out equal to n, proving our claim.

3.32. Let R (resp. B) denote the set of red (resp. black) squares in such a coloring, and for $s \in R \cup B$, let $f(s)n + g(s) + 1$ denote the number written in square s, where $0 \le f(s), g(s) \le n - 1$. Then it is clear that the value of $f(s)$ depends only on the row of s, while the value of $g(s)$ depends only on the column of s. Since every row contains exactly $n/2$ elements of R and $n/2$ elements of B,

$$\sum_{s \in R} f(s) = \sum_{s \in B} f(s).$$

Similarly, because every column contains exactly $n/2$ elements of R and $n/2$ elements of B,

$$\sum_{s \in R} g(s) = \sum_{s \in B} g(s).$$

It follows that

$$\sum_{s \in R} f(s)n + g(s) + 1 = \sum_{s \in B} f(s)n + g(s) + 1,$$

as desired.

3.33. The answer is only 101.

Each of the given numbers can be written

$$1 + 100 + 100^2 + \cdots + 100^n = p_n(10^2)\,,$$

where $p_n(x) = 1 + x + x^2 + \cdots + x^n$, $n = 1, 2, 3, \dots$. We have $(x-1)P_n(x) = x^{n+1} - 1$. If we set $x = 10^2$, we get

$$99 \cdot p_n(10^2) = 10^{2(n+1)} - 1 = (10^{n+1} - 1)(10^{n+1} + 1)\,.$$

If $p_n(10^2)$ is prime it must divide one of the factors of the RHS. It cannot divide $10^{n+1} - 1$ because this factor is less than $p_n(10^2)$, so $p_n(10^2)$ must divide the other factor. Hence $10^{n+1} - 1$ must divide 99. This is impossible for $n \ge 2$. In only remains to check the case $n = 1$. In this case we have $p_1(10^2) = 101$, which is prime.

3.34. By contradiction. Assume n divides $2^n - 1$ (note that this implies that n is odd). Let p be the smallest prime divisor of n, and let $n = p^k m$, where p does not divide m. Since n is odd we have that $p \ne 2$. By Fermat's Little Theorem we have

$2^{p-1} \equiv 1 \pmod p$. Also by Fermat's Little Theorem, $(2^{mp^{k-1}})^{p-1} \equiv 1 \pmod p$, hence $2^n = 2^{p^k m} = (2^{p^{k-1}m})^{p-1} \cdot 2^{p^{k-1}m} \equiv 2^{p^{k-1}m} \pmod p$. Repeating the argument we get $2^n = 2^{p^k m} \equiv 2^{p^{k-1}m} \equiv 2^{p^{k-2}m} \equiv \cdots \equiv 2^m \pmod p$. Since by hypothesis $2^n \equiv 1 \pmod p$, we have that $2^m \equiv 1 \pmod p$.
Next we use that if $2^a \equiv 1 \pmod p$, and $2^b \equiv 1 \pmod p$, then $2^{\gcd(a,b)} \equiv 1 \pmod p$. If $g = \gcd(n, p-1)$, then we must have $2^g \equiv 1 \pmod p$. But since p is the smallest prime divisor of n, and all prime divisors of $p-1$ are less than p, we have that n and p do not have common prime divisors, so $g = 1$, and consequently $2^g = 2$, contradicting $2^g \equiv 1 \pmod p$.

3.35. In spite of its apparent complexity this problem is very easy, because the digital root of b_n becomes a constant very quickly. First note that the digital root of a number a is just the reminder r of a modulo 9, and the digital root of a^n will be the remainder of r^n modulo 9.
For $a_1 = 31$ we have
digital root of a_1 = digital root of $31 = 4$;
digital root of a_1^2 = digital root of $4^2 = 7$;
digital root of a_1^3 = digital root of $4^3 = 1$;
digital root of a_1^4 = digital root of $4^4 = 4$;
and from here on it repeats with period 3, so the digital root of a_1^n is 1, 4, and 7 for remainder modulo 3 of n equal to 0, 1, and 2 respectively.
Next, we have $a_2 = 314 \equiv 2 \pmod 3$, $a_2^2 \equiv 2^2 \equiv 1 \pmod 3$, $a_2^3 \equiv 2^3 \equiv 2 \pmod 3$, and repeating with period 2, so the reminder of a_2^n depends only on the parity of n, with $a_2^n \equiv 1 \pmod 3$ if n is even, and $a_2^n \equiv 2 \pmod 3$ if n is odd.
And we are done because a_3 is odd, and the exponent of a_2 in the power tower defining b_n for every $n \geq 3$ is odd, so the reminder modulo 3 of the exponent of a_1 will be 2, and the reminder modulo 9 of b_n will be 7 for every $n \geq 3$.
Hence, the answer is 7.

4.1. If $x = \sqrt{2} + \sqrt{5}$ then

$$\begin{aligned} x^2 &= 7 + 2\sqrt{10}\,, \\ x^2 - 7 &= 2\sqrt{10}\,, \\ (x^2-7)^2 &= 40\,, \\ x^4 - 14x^2 + 9 &= 0\,. \end{aligned}$$

Hence the desired polynomial is $x^4 - 14x^2 + 9$.

4.2. We have that $p(x)+1$ has zeros at a, b, and c, hence $p(x)+1 = (x-a)(x-b)(x-c)q(x)$. If p had an integral zero d we would have

$$(d-a)(d-b)(d-c)q(d) = 1\,,$$

where $d-a$, $d-b$, and $d-c$ are distinct integers. But that is impossible, because 1 has only two possible factors, 1 and -1.

4.3. We prove it by showing that the sum is the root of a monic polynomial but not an integer—so by the rational roots theorem it must be irrational.
First we notice that $n < \sqrt{n^2+1} < n + 1/n$, hence the given sum is of the form

$$S = 1001 + \theta_1 + 1002 + \theta_2 + \cdots + 2000 + \theta_{1000}$$

were $0 < \theta_i < 1/1001$, consequently

$$0 < \theta_1 + \theta_2 + \cdots + \theta_{1000} < 1\,,$$

so S is not an integer.
Now we must prove that S is the root of a monic polynomial. More generally we will prove that a sum of the form

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}$$

where the a_i's are positive integers, is the root of a monic polynomial.[2] This can be proved by induction on n. For $n = 1$, $\sqrt{a_1}$ is the root of the monic polynomial $x^2 - a_1$. Next assume that $y = \sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}$ is a zero of a monic polynomial $P(x) = x^r + c_{r-1}x^{r-1} + \cdots + c_0$. We will find a polynomial that has $z = y + \sqrt{a_{n+1}}$ as a zero. We have

$$0 = P(y) = P(z - \sqrt{a_{n+1}}) = (z - \sqrt{a_{n+1}})^r + c_{r-1}(z - \sqrt{a_{n+1}})^{r-1} + \cdots + c_0\,.$$

Expanding the parentheses and grouping the terms that contain $\sqrt{a_{n+1}}$:

$$0 = P(z - \sqrt{a_{n+1}}) = z^r + Q(z) + \sqrt{a_{n+1}}\,R(z)\,.$$

Putting radicals on one side and squaring

$$(z^r + Q(z))^2 = a_{n+1}\,(R(z))^2\;,$$

so

$$T(x) = (x^r + Q(x))^2 - a_{n+1}\,(R(x))^2$$

is a monic polynomial with z as a root.

4.4. Consider the following polynomial:

$$Q(x) = (x+1)P(x) - x\,.$$

We have that $Q(k) = 0$ for $k = 0, 1, 2, \dots, n$, hence, by the *Factor theorem*,

$$Q(x) = Cx(x-1)(x-2)\dots(x-n)\,,$$

where C is a constant to be determined. Plugging $x = -1$ we get

$$Q(-1) = C(-1)(-2)\cdots(-(n+1))\,.$$

[2] The result can be obtained also by resorting to a known theorem on *algebraic integers* ("algebraic integer" is the mathematical term used to designate a root of a monic polynomial.) It is known that algebraic integers form a mathematical structure called *ring*, basically meaning that the sum, difference and product of two algebraic integers is an algebraic integer. Now, if a_i and k_i are positive integers, then $\sqrt[k_i]{a_i}$ is an algebraic integer, because it is a root of the monic polynomial $x^{k_i} - a_i$. Next, since the sum or difference of algebraic integers is an algebraic integer then $\pm \sqrt[k_1]{a_1} \pm \sqrt[k_2]{a_2} \pm \cdots \pm \sqrt[k_n]{a_n}$ is in fact an algebraic integer (note that the roots do not need to be square roots, and the signs can be combined in any way.)

On the other hand $Q(-1) = 0 \cdot P(-1) - (-1) = 1$, hence $C = \frac{(-1)^{n+1}}{(n+1)!}$.
Next, plugging in $x = n+1$ we get

$$(n+2)P(n+1) - (n+1) = C(n+1)! = \frac{(-1)^{n+1}}{(n+1)!}(n+1)! = (-1)^{n+1}\,,$$

hence

$$\boxed{P(n+1) = \frac{n+1+(-1)^{n+1}}{n+2}}\,.$$

4.5. Let the zeros be a, b, c, d. The relationship between zeros and coefficients yields

$$\begin{aligned} a+b+c+d &= 18 \\ ab+ac+ad+bc+bd+cd &= k \\ abc+abd+acd+bcd &= -200 \\ abcd &= -1984\,. \end{aligned}$$

Assume $ab = -32$ and let $u = a+b$, $v = c+d$, $w = cd$. Then

$$\begin{aligned} u+v &= 18 \\ -32+uv+w &= k \\ -32v+uw &= -200 \\ -32w &= -1984\,. \end{aligned}$$

From the last equation we get $w = 62$, and replacing in the other equations we easily get $u = 4$, $v = 14$. Hence

$$k = -32 + 4 \cdot 14 + 62 = \boxed{86}\,.$$

4.6. Let $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. The $(n-1)$-degree polynomial $p(x) - p(-x) = 2(a_1x + a_3x^3 + \cdots + a_{n-1}x^{n-1})$ vanishes at n different points, hence, it must be identically null, i.e., $a_1 = a_3 = \cdots = a_{n-1} = 0$. Hence $p(x) = a_0 + a_2x^2 + a_4x^4 \cdots + a_nx^n$, and $q(x) = a_0 + a_2x + a_4x^2 \cdots + a_nx^{n/2}$.

4.7. If k is an even integer we have $p(k) \equiv p(0) \pmod 2$, and if it is odd then $p(k) \equiv p(1) \pmod 2$. Since $p(0)$ and $p(1)$ are odd we have $p(k) \equiv 1 \pmod 2$ for every integer k, so $p(k)$ cannot be zero.

4.8. We must prove that $P(1) = 0$. Consider the four complex numbers $\rho_k = e^{2\pi ik/5}$, $k = 1, 2, 3, 4$. All of them verify $\rho_k^5 = 1$, so together with 1 they are the roots of $x^5 - 1$. Since $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$ then the ρ_k's are the roots of $x^4 + x^3 + x^2 + x + 1$. So

$$P(1) + \rho_k Q(1) + \rho_k^2 R(1) = 0\,.$$

Adding for $k = 1, 2, 3, 4$ and taking into account that the numbers ρ_k^2 are the ρ_k's in a different order we get

$$P(1) = 0\,.$$

4.9. The answer is no. We prove it by contradiction. Assume $(x-a)(x-b)(x-c)-1 = p(x)q(x)$, where p is linear and q is quadratic. Then $p(a)q(a) = p(b)q(b) = p(c)q(c) = -1$. If the coefficients of p and q must integers they can take only integer values, so in each product one of the factor must be 1 and the other one is -1. Hence either $p(x)$ takes the value 1 twice or it takes the value -1 twice. But a 1st degree polynomial cannot take the same value twice.

4.10. We prove it by contradiction. Suppose $g(x) = h(x)k(x)$, where $h(x)$ and $k(x)$ are non-constant polynomials with integral coefficients. Since $g(x) > 0$ for every x, $h(x)$ and $k(x)$ cannot have real roots, so they cannot change signs and we may suppose $h(x) > 0$ and $k(x) > 0$ for every x. Since $g(p_i) = 1$ for $i = 1, \dots, n$, we have $h(p_i) = k(p_i) = 1$, $i = 1, \dots, n$. If either $h(x)$ or $k(x)$ had degree less than n, it would constant, against the hypothesis, so they must be of degree n. Also we know that $h(x) - 1$ and $k(x) - 1$ are zero for $x = p_i$, $i = 1, \dots, n$, so their roots are precisely $p_1, \dots, p_n$, and we can write

$$\begin{aligned} h(x) &= 1 + a(x-p_1)\cdots(x-p_n) \\ k(x) &= 1 + b(x-p_1)\cdots(x-p_n)\,, \end{aligned}$$

where a and b are integers. So we have

$$(x-p_1)^2\cdots(x-p_n)^2 + 1 = \\ 1 + (a+b)(x-p_1)\cdots(x-p_n) + ab(x-p_1)^2\cdots(x-p_n)^2\,.$$

Hence

$$\begin{cases} a+b &= 0 \\ ab &= 1\,, \end{cases}$$

which is impossible, because there are no integers a, b verifying those equations.

4.11. Assume the quotient is $q(x)$ and the remainder is $r(x) = ax^2 + bx + c$. Then

$$x^{81} + x^{49} + x^{25} + x^9 + x = q(x)(x^3 - x) + r(x)\,,$$

Plugging in the values $x = -1, 0, 1$ we get $r(-1) = -5$, $r(0) = 0$, $r(1) = 5$. From here we get $a = c = 0$, $b = 5$, hence the remainder is $r(x) = 5x$.

4.12. For positive integer n we have $f(n) = \frac{n}{n+1}f(n-1) = \frac{n-1}{n+1}f(n-2) = \cdots = 0\cdot f(-1) = 0$. Hence $f(x)$ has infinitely many zeros, and must be identically zero $f(x) \equiv 0$.

4.13. By contradiction. Assume $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ have integral coefficients and degree less than 105. Let $\alpha_1, \dots, \alpha_k$ the (complex) roots of $g(x)$. For each $j = 1, \dots, k$ we have $\alpha_j^{105} = 9$, hence $|\alpha_j| = \sqrt[105]{9}$, and $|\alpha_1\alpha_2\cdots\alpha_k| = (\sqrt[105]{9})^k =$ the absolute value of the constant term of $g(x)$ (an integer.) But $(\sqrt[105]{9})^k = \sqrt[105]{3^{2k}}$ cannot be an integer.

4.14. The answer is $p = 5$. By Sophie Germain's Identity we have

$$x^4 + 4y^4 = (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy) = [(x+y)^2 + y^2][(x-y)^2 + y^2]\,,$$

which can be prime only if $x = y = 1$.

4.15. We have that a, b, c, d are distinct roots of $P(x) - 5$, hence

$$P(x) - 5 = g(x)(x - a)(x - b)(x - c)(x - d)\,,$$

where $g(x)$ is a polynomial with integral coefficients. If $P(k) = 8$ then

$$g(x)(x - a)(x - b)(x - c)(x - d) = 3\,,$$

but 3 is a prime number, so all the factors on the left but one must be ± 1. So among the numbers $(x - a)$, $(x - b)$, $(x - c)$, $(x - b)$, there are either two 1's or two -1's, which implies that a, b, c, d cannot be all distinct, a contradiction.

4.16. Calling $A_{n-1} = 1 + x + \cdots + x^{n-1}$, we have

$$\begin{aligned}
(1 + x + \cdots + x^n)^2 - x^n &= (A_{n-1} + x^n)^2 - x^n \\
&= A_{n-1}^2 + 2A_{n-1}x^n + x^{2n} - x^n \\
&= A_{n-1}^2 + 2A_{n-1}x^n + (x^n - 1)x^n \\
&= A_{n-1}^2 + 2A_{n-1}x^n + A_{n-1}(x - 1)x^n \\
&= A_{n-1}(A_{n-1} + 2x^n + (x - 1)x^n) \\
&= A_{n-1}(A_{n-1} + x^n + x^{n+1}) \\
&= (1 + x + \cdots + x^{n-1})(1 + x + \cdots + x^{n+1})\,.
\end{aligned}$$

4.17. Since $f(x)$ and $f(x) + f'(x)$ have the same leading coefficient, the limit of $f(x)$ as $x \to \pm\infty$ must be equal to that of $f(x) + f'(x)$, i.e., $+\infty$.
Note that f cannot have multiple real roots, because at any of those roots both f and f' would vanish, contradicting the hypothesis. So all real roots of f, if any, must be simple roots.
Since $f(x) \to +\infty$ for both $x \to \infty$ and $x \to -\infty$, it must have an even number of real roots (if any): $x_1 < x_2 < \cdots < x_{2n}$. Note that between x_1 and x_2, $f(x)$ must be negative, and by Rolle's theorem its derivative must be zero at some intermediate point $a \in (x_1, x_2)$, hence $f(a)+f'(a) = f(a) < 0$, again contradicting the hypothesis. Consequently, $f(x)$ has no real roots, and does not change sign at any point, which implies $f(x) > 0$ for all x.

4.18. This is a particular case of the well known Vandermonde determinant, but here we will find its value using arguments from polynomial theory. Expanding the determinant along the last column using Laplace formula we get

$$a_0(w, x, y) + a_1(w, x, y)z + a_2(w, x, y)z^2 + a_3(w, x, y)z^3\,,$$

where $a_i(w, x, y)$ is the cofactor of z^i.
Since the determinant vanishes when two columns are equal, that polynomial in z has zeros at $z = w$, $z = x$, $z = y$, hence it must be of the form

$$a_3(w, x, y, z)(z - y)(z - x)(z - w)\,.$$

Note that $a_3(w,x,y) = \begin{vmatrix} 1 & 1 & 1 \\ w & x & y \\ w^2 & x^2 & y^2 \end{vmatrix}$, which can be computed in an analogous way:

$$a_3(w,x,y) = b_2(w,x)(y-x)(y-w)\,,$$
$$b_2(w,x) = c_1(w)(x-w)\,,$$
$$c_1(w) = 1\,.$$

Hence the value of the given determinant is

$$(z-y)(z-x)(z-w)(y-x)(y-w)(x-w)\,.$$

4.19. Expanding the determinant along the last column using Laplace formula we get

$$a_0(w,x,y) + a_1(w,x,y)z + a_2(w,x,y)z^2 + a_4(w,x,y)z^4\,,$$

where $a_i(w,x,y)$ is the cofactor of z^i. In particular $a_4(w,x,y) = (y-x)(y-w)(x-w)$ by Vandermonde formula.
Since the determinant vanishes when two columns are equal, that polynomial in z has zeros at $z=w$, $z=x$, $z=y$, hence it must be of the form

$$a_4(w,x,y,z)(z-y)(z-x)(z-w)\,b(w,x,y,z) = \\ (z-y)(z-x)(z-w)(y-x)(y-w)(x-w)\,b(w,x,y,z)\,,$$

where $b(w,x,y,z)$ is some first degree homogeneous polynomial in w,x,y,z. Note that the value of $b(w,x,y,z)$ won't change by permutations of its arguments, so $b(w,x,y,z)$ is symmetric, and all its coefficients must be equal, hence $b(w,x,y,z) = k\cdot(w+x+y+z)$ for some constant k. The value of k can be found by computing the determinant for particular values of w,x,y,z, say $w=0$, $x=1$, $y=2$, $z=3$, and we obtain $k=1$. Hence the value of the determinant is

$$(z-y)(z-x)(z-w)(y-x)(y-w)(x-w)(w+x+y+z)\,.$$

4.20. The answer is no.
We can write the condition in matrix form in the following way:

$$\begin{pmatrix} 1 & x & x^2 \end{pmatrix} \begin{pmatrix} 1 \\ y \\ y^2 \end{pmatrix} = \begin{pmatrix} a(x) & c(x) \end{pmatrix} \begin{pmatrix} b(y) \\ d(y) \end{pmatrix}\,.$$

By assigning values $x=0,1,2$, $y=0,1,2$, we obtain the following identity:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} a(0) & c(0) \\ a(1) & c(1) \\ a(2) & c(2) \end{pmatrix} \begin{pmatrix} b(0) & b(1) & b(2) \\ d(0) & d(1) & d(2) \end{pmatrix}\,.$$

The product on the left hand side yields a matrix of rank 3, while the right hand side has rank at most 2, contradiction.

4.21. The answer is $P(x) = x$.
In order to prove this we show that $P(x)$ equals x for infinitely many values of x. In fact, let a_n the sequence $0, 1, 2, 5, 26, 677, \dots$, defined recursively $a_0 = 0$, and $a_{n+1} = a_n^2 + 1$ for $n \geq 0$. We prove by induction that $P(a_n) = a_n$ for every $n = 0, 1, 2, \dots$. In the basis case, $n = 0$, we have $P(0) = 0$. For the induction step assume $n \geq 1$, $P(a_n) = a_n$. Then $P(a_{n+1}) = P(a_n^2+1) = P(a_n)^2+1 = a_n^2+1 = a_{n+1}$. Since in fact $P(x)$ coincides with x for infinitely many values of x, we must have $P(x) = x$ identically.

4.22. Given a line $y = mx + b$, the intersection points with the given curve can be computed by solving the following system of equations

$$\begin{cases} y = 2x^4 + 7x^3 + 3x - 5 \\ y = mx + b\,. \end{cases}$$

Subtracting we get $2x^4 + 7x^3 + 3(x - m) - 5 - b = 0$. If the line intersects the curve in four different points, that polynomial will have four distinct roots x_1, x_2, x_3, x_4, and their sum will be minus the coefficient of x^3 divided by the coefficient of x^4, i.e., $-7/2$, hence

$$\frac{x_1 + x_2 + x_3 + x_4}{4} = -\frac{7}{8}\,.$$

4.23. The answer is $k = 3$, and an example is

$$(x + 2)(x + 1)x(x - 1)(x - 2) = x^5 - 5x^3 + 4x\,,$$

where $\{-2, -1, 0, 1, 2\}$ is the desired set of integers.
To complete the argument we must prove that k cannot be less than 3. It cannot be 1 because in that case the polynomial would be x^5, with all its five roots equal zero (note that a, b, c, d, e are the roots of the polynomial, and by hypothesis they must be distinct integers). Assume now that $k = 2$. Then the polynomial would be of the form $x^5 + nx^i = x^i(x^{5-i} + n)$, n a nonzero integer, $0 \leq i \leq 4$. If $i \geq 2$ then the polynomial would have two or more roots equal zero, contradicting the hypothesis. If $i = 1$ then the roots of the polynomial would be 0, and the roots of $x^4 + n$, at least two of which are non-real complex roots. If $i = 0$ then the polynomial is $x^5 + n$, which has four non-real complex roots.

4.24. The zeros of $x^4 - 13x^2 + 36$ are $x = \pm 2$ and ± 3, hence the condition is equivalent to $x \in [-3, -2] \cup [2, 3]$. On the other hand $f'(x) = 3x^2 - 3$, with zeros at $x = \pm 1$. This implies that $f(x)$ is monotonic on $[-3, -2]$ and $[2, 3]$, hence (with the given constrain) its maximum value can be attained only at the boundaries of those intervals. Computing $f(-3) = -18$, $f(-2) = -2$, $f(2) = 2$, $f(3) = 18$, we get that the desired maximum is 18.

4.25. Note that if $r(x)$ and $s(x)$ are any two functions, then

$$\max(r, s) = (r + s + |r - s|)/2.$$

Therefore, if $F(x)$ is the given function, we have

$$\begin{aligned} F(x) &= \max\{-3x-3,0\} - \max\{5x,0\} + 3x + 2 \\ &= (-3x-3+|3x-3|)/2 \\ &\qquad - (5x+|5x|)/2 + 3x + 2 \\ &= |(3x-3)/2| - |5x/2| - x + \frac{1}{2}, \end{aligned}$$

so we may set $f(x) = (3x-3)/2$, $g(x) = 5x/2$, and $h(x) = -x + \frac{1}{2}$.

4.26. Writing the given sums of powers as functions of the elementary symmetric polynomials of α, β, γ, we have

$$\begin{array}{lcl} \alpha+\beta+\gamma &=& s\,, \\ \alpha^2+\beta^2+\gamma^2 &=& s^2-2q\,, \\ \alpha^3+\beta^3+\gamma^3 &=& s^3-3qs+3p\,, \end{array}$$

where $s = \alpha+\beta+\gamma$, $q = \alpha\beta+\beta\gamma+\alpha\gamma$, $p = \alpha\beta\gamma$.
So we have $s = 2$, and from the second given equation get $q = -5$. Finally from the third equation we get $p = -7$. So, this is the answer, $\alpha\beta\gamma = -7$.
(Note: this is not needed to solve the problem, but by solving the equation $x^3 - sx^2 + qs - p = x^3 - 2x^2 + 5x + 7 = 0$ we find that the three numbers α, β, and γ are approx. 2.891954442, -2.064434534, and 1.172480094.)

4.27. Let $\alpha = (2+\sqrt{5})^{1/3} - (-2+\sqrt{5})^{1/3}$. By raising to the third power, expanding and simplifying we get that α verifies the following polynomial equation:

$$\alpha^3 + 3\alpha - 4 = 0\,.$$

We have $x^3 + 3x - 4 = (x-1)(x^2+x+4)$. The second factor has no real roots, hence $x^3 + 3x - 4$ has only one real root equal to 1, i.e., $\alpha = 1$.

4.28. The answer is affirmative, B can in fact guess the polynomial—call it $f(x) = a_0 + a_1x^2 + a_2x^2 + \cdots a_nx^n$. By asking A to evaluate it at 1, B gets an upper bound $f(1) = a_0 + a_1 + a_2 + \cdots a_n = M$ for the coefficients of the polynomial. Then, for any integer $N > M$, the coefficients of the polynomial are just the digits of $f(N) = a_0 + a_1N^2 + a_2N^2 + \cdots a_nN^n$ in base N.

4.29. If $f(x_0) = 0$ at some point x_0, then by hypothesis we would have $f'(x_0) > 0$, and f would be (strictly) increasing at x_0. This implies:
(1) If the polynomial f becomes zero at some point x_0, then $f(x) > 0$ for every $x > x_0$, and $f(x) < 0$ for every $x < x_0$.
Writing $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, we have $f(x) + f'(x) = a_nx^n + (a_{n-1} + na_n)x^{n-1} + \cdots + (a_0 + a_1)$. Given that that $f(x) + f'(x) > 0$ for all x, we deduce that $a_n = \lim_{n\to\infty} \frac{f(x)+f'(x)}{x^n} > 0$. On the other hand n must be even, otherwise $f(x) + f'(x)$ would become negative as $x \to -\infty$. Hence $f(x) > 0$ for $|x|$ large enough. By (1) this rules out the possibility of $f(x)$ becoming zero at some point x_0, and so it must be always positive.

Remark: The statement is not true for functions in general, e.g., $f(x) = -e^{-2x}$ verifies $f(x) + f'(x) = e^{-2x} > 0$, but $f(x) < 0$ for every x.

4.30. The answer is No. If $P(x)$ has two real roots we would have $b^2 > 4ac$. Analogously for $R(x)$ and $Q(x)$ we should have $a^2 > 4cb$, and $c^2 > 4ab$ respectively. Multiplying the inequalities we get $a^2b^2c^2 > 64a^2b^2c^2$, which is impossible.

4.31. First we prove (by contradiction) that $f(x)$ has no real roots. In fact, if x_1 is a real root of $f(x)$, then we have that $x_2 = x_1^2 + x_1 + 1$ is also a real root of $f(x)$, because $f(x_1^2 + x_1 + 1) = f(x_1)g(x_1) = 0$. But $x_1^2 + 1 > 0$, hence $x_2 = x_1^2 + x_1 + 1 > x_1$. Repeating the reasoning we get that $x_3 = x_2^2 + x_2 + 1$ is another root of $f(x)$ greater than x_2, and so on, so we get an infinite increasing sequence of roots of $f(x)$, which is impossible. Consequently $f(x)$ must have even degree, because all odd degree polynomials with real coefficients have at least one real root. Q.E.D.
Note: An example of a polynomial with the desired property is: $f(x) = x^2 + 1$, $f(x^2 + x + 1) = (x^2 + 1)(x^2 + 2x + 2)$.
Remark: The result is not generally true for polynomials with complex coefficients—counterexample: $f(x) = x + i$, $f(x^2 + x + 1) = x^2 + x + 1 + i = (x + i)(x + 1 - i)$.

4.32. By contradiction. We have that $a_0 = P(0)$ must be a prime number. Also, $P(ka_0)$ is a multiple of a_0 for every $k = 0, 1, 2, \dots$, but if $P(ka_0)$ is prime then $P(ka_0) = a_0$ for every $k \geq 0$. This implies that the polynomial $Q(x) = P(a_0x) - a_0$ has infinitely many roots, so it is identically zero, and $P(a_0x) = a_0$, contradicting the hypothesis that P is of degree at least 1.

5.1. If $m = a^2 + b^2$ and $n = c^2 + d^2$, then consider the product $z = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$. We have

$$|z|^2 = |a + bi|^2|c + di|^2 = (a^2 + b^2)(c^2 + d^2) = mn\,,$$

and

$$|z|^2 = (ac - bd)^2 + (ad + bc)^2\,,$$

so mn is also in fact a sum of two perfect squares.

5.2. The left hand side of the equality is the imaginary part of

$$\sum_{k=0}^{n} e^{ik} = \frac{e^{i(n+1)} - 1}{e^i - 1} = \frac{e^{i(n+1/2)} - e^{-i/2}}{e^{i/2} - e^{-i/2}} = \frac{\cos\left(n + \frac{1}{2}\right) - \cos\frac{1}{2} + i\{\sin\left(n + \frac{1}{2}\right) + \sin\frac{1}{2}\}}{2i\sin\frac{1}{2}}\,.$$

The imaginary part of that expression is

$$\frac{\cos\frac{1}{2} - \cos\left(n + \frac{1}{2}\right)}{2\sin\frac{1}{2}} = \frac{\sin\frac{n}{2}\sin\frac{n+1}{2}}{\sin\frac{1}{2}}$$

5.3. We have that $z = e^{\pm ia}$, so $z + 1/z = e^{ia} + e^{-ia} = 2\cos a$, hence:

$$z^n + 1/z^n = e^{ina} + e^{-ina} = 2\cos na\,.$$

5.4. Factoring a polynomial is easier to accomplish if we can find its roots. In this case we will look for roots that are roots of unity $e^{2k\pi i/n}$:

$$p(e^{2k\pi i/n}) = e^{10k\pi i/n} + e^{2k\pi i/n} + 1\,.$$

The three terms of that expression are complex numbers placed on the unit circle at the vertices of an equilateral triangle for $n = 3$ and $k = 1, 2$, so if $\omega = e^{2k\pi i/3}$, then ω and ω^2 are roots of $p(z)$, hence $p(z)$ is divisible by $(z-\omega)(z-\omega^2) = z^2+z+1$. By long division we find that the other factor is $z^3 - z^2 + 1$, hence:

$$p(z) = (z^2+z+1)(z^3-z^2+1)\,.$$

5.5. Write $\sin t = (e^{ti} - e^{-ti})/2i$ and consider the polynomial

$$p(x) = \prod_{k=1}^{n-1}(x - e^{2\pi ik/n})\,.$$

We have:

$$P = \prod_{k=1}^{n-1} \sin\frac{k\pi}{n} = \prod_{k=1}^{n-1} \frac{e^{\pi ik/n} - e^{-\pi ik/n}}{2i} = \frac{e^{-\pi i(n-1)/2}}{(2i)^{n-1}} \prod_{k=1}^{n-1}(e^{2\pi ik/n} - 1) = \frac{p(1)}{2^{n-1}}\,.$$

On the other hand the roots of $p(x)$ are all nth roots of 1 except 1, so $(x-1)p(x) = x^n - 1$, and

$$p(x) = \frac{x^n-1}{x-1} = 1 + x + x^2 + \cdots + x^{n-1}\,.$$

Consequently $p(1) = n$, and $P = \dfrac{n}{2^{n-1}}$.

5.6. Assume the vertices of the n-gon placed on the complex plane at the nth roots of unity $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$, where $\zeta = e^{2\pi i/n}$. Then the length of the diagonal connecting vertices j and k is $|\zeta^i - \zeta^k|$, and the desired product can be written

$$P = \prod_{0\le j<k<n} |\zeta^j - \zeta^k|\,.$$

By symmetry we obtain the same product if we replace the condition $j < k$ with $k < j$, and multiplying both expressions together we get:

$$P^2 = \prod_{\substack{0\le j,k<n\\ j\ne k}} |\zeta^j - \zeta^k| = |\zeta^j| \prod_{\substack{0\le j,k<n\\ j\ne k}} |1 - \zeta^{k-j}|\,.$$

Note that $|\zeta^j| = 1$, and for each k, $r = k-j$ takes all non-zero values from $k-n+1$ to k. Since $\zeta^r = \zeta^{r+n}$ we may assume that r ranges from 1 to $n-1$, so we can rewrite the product like this:

$$P^2 = \left(\prod_{r=1}^{n-1} |1-\zeta^r|\right)^n\,.$$

Next consider the polynomial

$$p(x) = \prod_{r=1}^{n-1} (x - \zeta^r) \,.$$

Its roots are the same roots of $x^n - 1$ except 1, hence $x^n - 1 = (x-1)p(x)$ and

$$p(x) = \frac{x^n - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{n-1} \,,$$

hence

$$\prod_{r=1}^{n-1} (1 - \zeta^r) = p(1) = n \,.$$

consequently $P^2 = n^n$, and $P = n^{n/2}$.

5.7. Define $h(x) = f(x) + ig(x)$. Then h is differentiable and $h'(0) = bi$ for some $b \in \mathbb{R}$. The given equations can be reinterpret as $h(x+y) = h(x)h(y)$. Differentiating respect to y and substituting $y = 0$ we get $h'(x) = h(x)h'(0) = bi \cdot h(x)$, so $h(x) = Ce^{bix}$ for some $C \in \mathbb{C}$. From $h(0+0) = h(0)h(0)$ we get $C = C^2$. If $C = 0$ then $h = 0$ and f and g would be constant, contradicting the hypothesis. Thus $C = 1$. Finally, for any $x \in \mathbb{R}$,

$$f(x)^2 + g(x)^2 = |h(x)|^2 = |e^{bix}|^2 = 1 \,.$$

5.8. Assume the lights placed on the complex plane at the nth roots of unity $1, \zeta, \zeta^2, \dots, \zeta^{n-1}$, where $\zeta = e^{2\pi i/n}$. Without loss of generality we may assume that the light at 1 is initially on. Now, if $d < n$ is a divisor of n and the lights $\zeta^a, \zeta^{a+d}, \zeta^{a+2d}, \dots, \zeta^{a+(\frac{n}{n}-1)d}$ have the same state, then we can change the state of this n/d lights. The sum of these is

$$\zeta^a + \zeta^{a+d} + \zeta^{a+2d} + \cdots + \zeta^{a+(\frac{n}{n}-1)d} = \zeta^a \left(\frac{1 - \zeta^n}{1 - \zeta^d} \right) = \zeta^a \left(\frac{1 - 1}{1 - \zeta^d} \right) = 0 \,.$$

So if we add up all the roots that are "on", the sum will never change. The original sum was 1, and the goal is to get all the lights turned on. That sum will be

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = \frac{1 - \zeta^n}{1 - \zeta} = 0 \neq 1 \,.$$

Hence we can never turn on all the lights.

5.9. Let $z_1 = a - bi$, $z_2 = u + vi$. Then $|z_1|^2 = a^2 + b^2$, $|z_2| = u^2 + v^2$, $\Re(z_1 z_2) = au + bv$, $\Im(z_1 z_2) = 1$. On the other hand:

$$|z_1 z_2|^2 = \Re(z_1 z_2)^2 + \Im(z_1 z_2)^2 = \Re(z_1 z_2)^2 + 1 \,.$$

Now for any real t,

$$(t\sqrt{3} + 1)^2 \geq 0 \implies 3t^2 + 1 \geq -2t\sqrt{3} \implies 4t^2 + 4 \geq (\sqrt{3} - t)^2 \,.$$

Hence

$$(|z_1|^2 + |z_2|^2)^2 \geq 4|z_1 z_2|^2 = 4\big(\Re(z_1 z_2)^2 + 1\big) \geq \big(\sqrt{3} - \Re(z_1 z_2)\big)^2 \,.$$

So, $|z_1|^2 + |z_2|^2 \geq \sqrt{3} - \Re(z_1 z_2)$. Or $|z_1|^2 + |z_2|^2 + \Re(z_1 z_2) \geq \sqrt{3}$, as required.

6.1. We have

$$\binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n = (1+x)^n .$$

Differentiating respect to x:

$$\binom{n}{1} + 2\binom{n}{2} x + 3\binom{n}{3} x^2 + \cdots + n\binom{n}{n} = n(1+x)^{n-1} .$$

Plugging in $x = 1$ we get the desired identity

6.2. The desired expression states the equality between the coefficient of x^n in each of the following expansions:

$$(1+x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k ,$$

and

$$\{(1+x)^n\}^2 = \left\{ \sum_{k=0}^{n} \binom{n}{k} x^k \right\}^2 = \sum_{k=0}^{n} \sum_{i+j=k} \binom{n}{i}\binom{n}{j} x^k .$$

Taking into account that $\binom{n}{j} = \binom{n}{n-j}$, for $k = n$ we get

$$\sum_{i+j=n} \binom{n}{i}\binom{n}{j} = \sum_{i+j=n} \binom{n}{i}\binom{n}{n-j} = \sum_{i=1}^{n} \binom{n}{i}^2 ,$$

and that must be equal to the coefficient of x^n in $(1+x)^{2n}$, which is $\binom{2n}{n}$.

6.3. This is just a generalization of the previous problem. We have

$$(1+x)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k ,$$

and

$$\begin{aligned}(1+x)^m(1+x)^n &= \left\{ \sum_{i=0}^{m} \binom{m}{i} x^i \right\} \left\{ \sum_{j=0}^{n} \binom{n}{j} x^j \right\} \\ &= \sum_{k=0}^{m+n} \sum_{\substack{i+j=k \\ 0\leq i,j\leq k}} \binom{n}{j}\binom{m}{i} x^k .\end{aligned}$$

The coefficient of x^k must be the same on both sides, so:

$$\binom{m+n}{k} = \sum_{\substack{i+j=k \\ 0\leq i,j\leq k}} \binom{n}{j}\binom{m}{i} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} ,$$

where we replace $i = k - j$ in the last step.

6.4. The generating function for the Fibonacci sequence is

$$0 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots = \frac{x}{1 - x - x^2}\,.$$

The desired sum is the left hand side with $x = 1/2$, hence its value is

$$0 + \frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \cdots = \frac{\frac{1}{2}}{1 - \frac{1}{2} - \frac{1}{2^2}} = \boxed{2}\,.$$

6.5. The generating function of u_n is the following:

$$\begin{aligned} f(x) &= u_0 + u_1 x + u_2 x^2 + \cdots \\ &= (1 + x^2 + x^4 + x^6 + \cdots)(1 + x^5 + x^{10} + x^{15} + \cdots) \\ &= \frac{1}{1 - x^2}\frac{1}{1 - x^5} \\ &= \frac{1}{1 - x^2 - x^5 + x^7}\,. \end{aligned}$$

Hence

$$1 = (1 - x^2 - x^5 + x^7)(u_0 + u_1 x + u_2 x^2 + \cdots)\,.$$

From here we get that $1 \cdot u_0 = 1$, hence $u_0 = 1$. Similarly $1 \cdot u_1 = 0$, hence $u_1 = 0$, etc., so we get $u_0 = u_2 = u_4 = u_5 = u_7 = 1$, and $u_1 = u_3 = 0$. Then for $k > 7$ the coefficient of x^k of the product must be

$$u_k - u_{k-2} - u_{k-5} + u_{k-7} = 0\,.$$

So we get the following recursive relation for the terms of the sequence:

$$u_k = u_{k-2} + u_{k-5} - u_{k-7}\,,$$

together with the initial conditions $u_0 = u_2 = u_4 = u_5 = u_7 = 1$, and $u_1 = u_3 = 0$.

6.6. The answer equals the coefficient of x^{10} in the expansion of

$$(1 + x + x^2 + \cdots + x^9)^6\,.$$

Since $1+x+x^2+\cdots = 1/(1-x)$ the answer can be obtained also from the coefficient of x^{10} in the Maclaurin series of $1/(1-x)^6 = (1-x)^{-6}$. Since that includes six sequences of the form $0, 0, \cdots, 10, \cdots, 0$ we need to subtract 6, so the final answer is

$$\begin{aligned} \binom{-6}{10} - 6 &= \frac{(-6)(-7)(-8)(-9)(-10)(-11)(-12)(-13)(-14)(-15)}{10!} - 6 \\ &= 3003 - 6 = \boxed{2997}\,. \end{aligned}$$

6.7. Consider the polynomial $P(x) = a_1 + a_2 x + a_3 x^2 + \cdots + a_{50} x^{49}$. If r is a 3rd root of unity different from 1 then $P(r) = c(1 + r + r^2)$, where $c = a_k + a_{k+3} + a_{k+6} + \cdots$ But $1 + r + r^2 = (r^3 - 1)/(r - 1) = 0$, so $P(r) = 0$. Analogous reasoning shows that $P(r) = 0$ for each 5th, 7th, 11th, 13th, 17th root of unity r different from 1. Since there are respectively $2 + 4 + 6 + 10 + 12 + 16 = 50$ such roots of unity we have that $P(r)$ is zero for 50 different values of r. But a 49-degree polynomial has only 49 roots, so $P(x)$ must be identically zero.

7.1. Let $f(n)$ be that number. Then we easily find $f(0) = 1$, $f(1) = 2$, $f(2) = 3$, $f(3) = 5, \dots$ suggesting that $f(n) = F_{n+2}$ (shifted Fibonacci sequence). We prove this by showing that $f(n)$ verifies the same recurrence as the Fibonacci sequence. The subsets of $\{1, 2, \dots, n\}$ that contain no two consecutive elements can be divided into two classes, the ones not containing n, and the ones containing n. The number of the ones not containing n is just $f(n-1)$. On the other hand the ones containing n cannot contain $n-1$, so their number equals $f(n-2)$. Hence $f(n) = f(n-1) + f(n-2)$, QED.

7.2. Let x_n be the number of regions in the plane determined by n "vee"s. Then $x_1 = 2$, and $x_{n+1} = x_n + 4n + 1$. We justify the recursion by noticing that the $(n+1)$th "vee" intersects each of the other "vee"s at 4 points, so it is divided into $4n + 1$ pieces, and each piece divides one of the existing regions of the plane into two, increasing the total number of regions by $4n + 1$. So the answer is

$$x_n = 2 + (4 + 1) + (4 \cdot 2 + 1) + \cdots + (4 \cdot (n-1) + 1) = 2n^2 - n + 1\,.$$

7.3. Let x_n be the number of tilings of an $n \times 2$ rectangle by dominoes. We easily find $x_1 = 1$, $x_2 = 2$. For $n \geq 3$ we can place the rightmost domino vertically and tile the rest of the rectangle in x_{n-1} ways, or we can place two horizontal dominoes to the right and tile the rest in x_{n-2} ways, so $x_n = x_{n-1} + x_{n-2}$. So the answer is the shifted Fibonacci sequence, $x_n = F_{n+1}$.

7.4. Let f_n denote the number of minimal selfish subsets of $\{1, 2, \dots, n\}$. For $n = 1$ we have that the only selfish set of $\{1\}$ is $\{1\}$, and it is minimal. For $n = 2$ we have two selfish sets, namely $\{1\}$ and $\{1, 2\}$, but only $\{1\}$ is minimal. So $f_1 = 1$ and $f_2 = 1$. For $n > 2$ the number of minimal selfish subsets of $\{1, 2, \dots, n\}$ not containing n is equal to f_{n-1}. On the other hand, for any minimal selfish set containing n, by removing n from the set and subtracting 1 from each remaining element we obtain a minimal selfish subset of $\{1, 2, \dots, n\}$. Conversely, any minimal selfish subset of $\{1, 2, \dots, n-2\}$ gives raise to a minimal selfish subset of $\{1, 2, \dots, n\}$ containing n by the inverse procedure. Hence the number of minimal selfish subsets of $\{1, 2, \dots, n\}$ containing n is f_{n-2}. Thus $f_n = f_{n-1} + f_{n-2}$, which together with $f_1 = f_2 = 1$ implies that $f_n = F_n$ (nth Fibonacci number.)

7.5. Assume that $b_1, b_2, \dots, b_n$ is a derangement of the sequence $a_1, a_2, \dots, a_n$. The element b_n can be any of $a_1, \dots, a_{n-1}$, so there are $n - 1$ possibilities for its value. Once we have fixed the value of $b_n = a_k$ for some $k = 1, \dots, n-1$, the derangement can be of one of two classes: either $b_k = a_n$, or $b_k \neq a_n$. The first class coincides with the derangements of $a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_{n-1}$, and there are D_{n-2} of them. The second class coincides with the derangements of $a_1, \dots, a_{n-1}$ with a_k replaced with a_n, and there are D_{n-1} of them.

7.6. We have that α and β are the roots of the polynomial

$$(x-\alpha)(x-\beta) = x^2 - sx + p\,,$$

where $s = \alpha + \beta$, $p = \alpha\beta$.
We have that $s = a_1$ is an integer. Also, $2p = a_1^2 - a_2$ is an integer. The given sequence verifies the recurrence

$$a_{n+2} = s\,a_{n+1} - p\,a_n\,,$$

hence

$$2^{\lfloor \frac{n+1}{2} \rfloor} a_{n+2} = s\,2^{\lfloor \frac{n+1}{2} \rfloor} a_{n+1} - 2p\,2^{\lfloor \frac{n-1}{2} \rfloor} a_n\,.$$

From here we get the desired result by induction.

7.7. The general solution for the recurrence can be expressed using the roots of its characteristic polynomial

$$x^2 - \frac{10x}{3} + 1 = 0\,.$$

The roots are 3 and 1/3, hence a general solution is $x_n = A \cdot 3^n + B \cdot 3^{-n}$. If the sequence converges then $A = 0$, and the condition $x_0 = 18$ yields $B = 18$, hence the sequence is $x_n = 18 \cdot 3^{-n}$, the limit is 0, and $x_1 = 18/3 = 6$.

8.1. The solution is based on the fact that $\sqrt{u^2} = |u|$. Letting $u = 1 \pm \sqrt{x-1}$ we have that $u^2 = x \pm 2\sqrt{x-1}$, hence the given function turns out to be:

$$f(x) = |1 + \sqrt{x-1}| + |1 - \sqrt{x-1}|\,,$$

Defined for $x \geq 1$.
The expression $1 + \sqrt{x-1}$ is always positive, hence $|1 + \sqrt{x-1}| = 1 + \sqrt{x-1}$. On the other hand $|1 - \sqrt{x-1}| = 1 - \sqrt{x-1}$ if $1 - \sqrt{x-1} \geq 0$ and $|1 - \sqrt{x-1}| = -1 + \sqrt{x-1}$ if $1 - \sqrt{(x-1)} \leq 0$, hence

$$f(x) = \begin{cases} 1 + \sqrt{x-1} + 1 - \sqrt{x-1} = 2 & \text{if } 1 - \sqrt{x-1} \geq 0 \\ 1 + \sqrt{x-1} - 1 + \sqrt{x-1} = 2\sqrt{x-1} & \text{if } 1 - \sqrt{x-1} < 0 \end{cases}$$

So the function is equal to 2 if $1 - \sqrt{x-1} \geq 0$, which happens for $1 \leq x \leq 2$. So $f(x) = 2$ (constant) in $[1, 2]$.

8.2. The desired value is the limit of the following sequence:

$$a_1 = \sqrt{2}$$

$$a_2 = \sqrt{2 + \sqrt{2}}$$

$$a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$$

$$\ldots$$

defined by the recursion $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ $(n \geq 1)$.

First we must prove that the given sequence has a limit. To that end we prove

1. The sequence is bounded. More specifically, $0 < a_n < 2$ for every $n = 1, 2, \dots$. This can be proved by induction. It is indeed true for $a_1 = \sqrt{2}$. Next, if we assume that $0 < a_n < 2$, then $0 < a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 2} = \sqrt{4} = 2$.
2. The sequence is increasing. In fact: $a_{n+1}^2 = 2 + a_n > a_n + a_n = 2a_n > a_n^2$, hence $a_{n+1} > a_n$.

According to the *Monotonic Sequence Theorem*, every bounded monotonic (increasing or decreasing) sequence has a limit, hence a_n must have in fact a limit $L = \lim_{n\to\infty} a_n$.

Now that we know that the sequence has a limit L, by taking limits in the recursive relation $a_{n+1} = \sqrt{2 + a_n}$, we get $L = \sqrt{2 + L}$, hence $L^2 - L - 2 = 0$, so $L = 2$ or -1. Since $a_n > 0$ then $L \geq 0$, hence $L = 2$. Consequently:

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}} = 2\,.$$

8.3. We will prove that the answer is $(3 + \sqrt{5})/2$.

The value of the infinite continued fraction is the limit L of the sequence defined recursively $x_0 = 2207$, $x_{n+1} = 2207 - 1/x_n$, which exists because the sequence is decreasing (induction). Taking limits in both sides we get that $L = 2007 - 1/L$. Since $x_n > 1$ for all n (also proved by induction), we have that $L \geq 1$. If we call the answer r we have $r^8 = L$, so $r^8 + 1/r^8 = 2207$. Then $(r^4 + 1/r^4)^2 = r^8 + 2 + 1/r^8 = 2 + 2207 = 2209$, hence $r^4 + 1/r^4 = \sqrt{2209} = 47$. Analogously, $(r^2 + 1/r^2)^2 = r^4 + 2 + 1/r^4 = 2 + 47 = 49$, so $r^2 + 1/r^2 = \sqrt{49} = 7$. And $(r + 1/r)^2 = r^2 + 2 + 1/r^2 = 2 + 7 = 9$, so $r + 1/r = \sqrt{9} = 3$. From here we get $r^2 - 3r + 1 = 0$, hence $r = (3 \pm \sqrt{5})/2$, but $r = L^{1/8} \geq 1$, so $r = (3 + \sqrt{5})/2$.

8.4. That function coincides with $g(x) = 1/(1 + x^2)$ at the points $x = 1/n$, and the derivatives of g at zero can be obtained from its Maclaurin series $g(x) = 1 - x^2 + x^4 - x^6 + \cdots$, namely $g^{(2k)}(0) = (-1)^k k!$ and $g^{(2k+1)}(0) = 0$. In order to prove that the result applies to f too we have to study their difference $h(x) = f(x) - g(x)$.

We have that $h(x)$ is infinitely differentiable. Also $h(1/n) = 0$ for $n = 1, 2, 3, \dots$, hence $h(0) = \lim_{n\to\infty} h(1/n) = 0$. By Rolle's theorem, $h'(x)$ has zeros between the zeros of $h(x)$, hence $h'(0)$ is the limit of a sequence of zeros, so $h'(0) = 0$. The same is true about all derivatives of h at zero. This implies that $f^{(k)}(0) = g^{(k)}(0)$ for every $k = 1, 2, 3, \dots$, hence $f^{(2k)}(0) = (-1)^k k!$ and $f^{(2k+1)}(0) = 0$.

8.5. By looking at the graph of the function $y = 1/x$ we can see that

$$\int_n^{2n} \frac{1}{x}\,dx < \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} < \int_{n-1}^{2n-1} \frac{1}{x}\,dx\,.$$

We have

$$\int_n^{2n} \frac{1}{x}\,dx = \ln(2n) - \ln n = \ln 2\,,$$

$$\int_{n-1}^{2n-1} \frac{1}{x}\,dx = \ln(2n-1) - \ln(n-1) = \ln\left\{\frac{2n-1}{n-1}\right\} \underset{n\to\infty}{\longrightarrow} \ln 2\,.$$

Hence by the Squeeze Theorem, the desired limit is $\ln 2$.

8.6. Let P be the limit. Then

$$\ln(P) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \ln\left(1 + \frac{k}{n}\right)$$

That sum is a Riemann sum for the following integral:

$$\int_0^1 \ln(1+x)\,dx = [(1+x)(\ln(1+x) - 1)]_0^1 = 2\ln 2 - 1\,.$$

Hence $P = e^{2\ln 2 - 1} = 4/e$.

8.7. The series on the left is $xe^{-x^2/2}$. Since the terms of the second sum are non-negative, we can interchange the sum and integral:

$$\int_0^\infty xe^{-x^2/2} \sum_{n=0}^\infty \frac{x^{2n}}{2^{2n}(n!)^2}\,dx = \sum_{n=0}^\infty \int_0^\infty xe^{-x^2/2} \frac{x^{2n}}{2^{2n}(n!)^2}\,dx$$

The term for $n = 0$ is

$$\int_0^\infty xe^{-x^2/2}\,dx = \left[-e^{-x^2/2}\right]_0^\infty = 0 - (-1) = 1\,.$$

Next, for $n \geq 1$, integrating by parts:

$$\int_0^\infty x^{2n}\left(xe^{-x^2/2}\right)dx = \underbrace{\left[-x^{2n}e^{-x^2/2}\right]_0^\infty}_{0} + 2n\int_0^\infty x^{2(n-1)}\left(xe^{-x^2/2}\right)dx\,.$$

Thus, by induction

$$\int_0^\infty x^{2n}\left(xe^{-x^2/2}\right)dx = 2\cdot 4\cdot 6 \cdots 2n\,.$$

Hence the integral is

$$\sum_{n=0}^\infty \frac{1}{2^n n!} = e^{1/2} = \sqrt{e}\,.$$

8.8. The answer is affirmative, in fact *any* real number r is the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$. First assume

$$\sqrt[3]{n} \leq r + \sqrt[3]{m} < \sqrt[3]{n+1}\,,$$

which can be accomplished by taking $n = \lfloor (r + \sqrt[3]{m})^3 \rfloor$. We have

$$\begin{aligned} 0 \le r - (\sqrt[3]{n} - \sqrt[3]{m}) &< \sqrt[3]{n+1} - \sqrt[3]{n} \\ &= \frac{1}{\sqrt[3]{(n+1)^2} + \sqrt[3]{(n+1)n} + \sqrt[3]{n^2}} . \end{aligned}$$

Since the last expression tends to 0 as $n \to \infty$, we have that

$$r = \lim_{m \to \infty} \left\{ \sqrt[3]{\lfloor (r + \sqrt[3]{m})^3 \rfloor} - \sqrt[3]{m} \right\} .$$

8.9. From $f(f(x)) = 1/f(x)$ we get that $f(y) = 1/y$ for all $y \in f(\mathbb{R})$. Hence $f(999) = 1/999$. Since f is continuous it takes all possible values between $1/999$ and 999, in particular $500 \in f(\mathbb{R})$. Hence $f(500) = 1/500$.

8.10. Consider the function $g : [0, 1998/1999] \to \mathbb{R}$, $g(x) = f(x) - f(x + 1/999)$. Then g is continuous on $[0, 1998/1999]$, and verifies

$$\sum_{k=0}^{1998} g(k/1999) = f(1) - f(0) = 0 .$$

Since the sum is zero it is impossible that all its terms are positive or all are negative, so either one is zero, or there are two consecutive terms with opposite signs. In the former case, $g(k/1999) = 0$ for some k, so $f(k/1999) = f((k+1)/1999)$ and we are done. Otherwise, if there are two consecutive terms $g(k/1999)$ and $g((k+1)/1999)$ with different signs, then for some $x \in [k/1999, (k+1)/1999]$ we have $g(x) = 0$, hence $f(x) = f(x + 1/1999)$, and we are also done.

8.11. The answer is $c \ge 1/2$.
In fact, the given inequality can be written like this:

$$e^{cx^2} - \frac{e^x + e^{-x}}{2} \ge 0 .$$

The Taylor expansion of the left hand side is

$$e^{cx^2} - \frac{e^x + e^{-x}}{2} = \left(c - \frac{1}{2}\right) x^2 + \left(\frac{c^2}{2!} - \frac{1}{4!}\right) x^4 + \left(\frac{c^3}{3!} - \frac{1}{6!}\right) x^6 + \cdots$$

We see that for $c \ge 1/2$ all the coefficients are non-negative, and the inequality holds.
On the other hand, if $c < 1/2$ we have

$$\lim_{x \to 0} \frac{e^{cx^2} - \frac{e^x + e^{-x}}{2}}{x^2} = c - \frac{1}{2} < 0 ,$$

so in a neighborhood of 0 the numerator must become negative, and the inequality does not hold.

8.12. There is no such sequence. If they were convergent their sum would be convergent too, but by the AM-GM inequality we have:

$$\sum_{n=1}^{\infty}\left(a_n+\frac{1}{n^2a_n}\right)\geq\sum_{n=1}^{\infty}\frac{2}{n}=\infty\,.$$

9.1. We divide the set into n-classes $\{1,2\},\{3,4\},\dots,\{2n-1,2n\}$. By the pigeonhole principle, given $n+1$ elements, at least two of them will be in the same class, $\{2k-1,2k\}$ $(1\leq k\leq n)$. But $2k-1$ and $2k$ are relatively prime because their difference is 1.

9.2. For each odd number $\alpha=2k-1$, $k=1,\dots,n$, let C_α be the set of elements x in S such that $x=2^i\alpha$ for some i. The sets $C_1,C_3,\dots,C_{2n-1}$ are a classification of S into n classes. By the pigeonhole principle, given $n+1$ elements of S, at least two of them will be in the same class. But any two elements of the same class C_α verify that one is a multiple of the other one.

9.3. The given set can be divided into 18 subsets $\{1\}$, $\{4,100\}$, $\{7,97\}$, $\{10,94\}$,..., $\{49,55\}$, $\{52\}$. By the pigeonhole principle two of the numbers will be in the same set, and all 2-element subsets shown verify that the sum of their elements is 104.

9.4. For $k=1,2,\dots,8$, look at the digit used in place k for each of the 4 given elements. Since there are only 3 available digits, two of the elements will use the same digit in place k, so they coincide at that place. Hence at each place, there are at least two elements that coincide at that place. Pick any pair of such elements for each of the 8 places. Since there are 8 places we will have 8 pairs of elements, but there are only $\binom{4}{2}=6$ two-element subsets in a 4-element set, so two of the pairs will be the same pair, and the elements of that pair will coincide in two different places.

9.5. Let a_j the number of games played from the 1st through the jth day of the month. Then $a_1,a_2,\dots,a_{30}$ is an increasing sequence of distinct positive integers, with $1\leq a_j\leq 45$. Likewise, $b_j=a_j+14$, $j=1,\dots,30$ is also an increasing sequence of distinct positive integers with $15\leq b_j\leq 59$. The 60 positive integers $a_1,\dots,a_{30},b_1,\dots,b_{30}$ are all less than or equal to 59, so by the pigeonhole principle two of them must be equal. Since the a_j's are all distinct integers, and so are the b_j's, there must be indices i and j such that $a_i=b_j=a_j+14$. Hence $a_i-a_j=14$, i.e., exactly 14 games were played from day $j+1$ through day i.

9.6. Let $\{x\}=x-\lfloor x\rfloor$ denote the fractional part of x. For $i=0,\dots,n$, put $s_i=x_1+\dots+x_i$ (so that $s_0=0$). Sort the numbers $\{s_0\},\dots,\{s_n\}$ into ascending order, and call the result $t_0,\dots,t_n$. Since $0=t_0\leq\dots\leq t_n<1$, the differences

$$t_1-t_0,\dots,t_n-t_{n-1},1-t_n$$

are nonnegative and add up to 1. Hence (as in the pigeonhole principle) one of these differences is no more than $1/(n+1)$; if it is anything other than $1-t_n$, it equals $\pm(\{s_i\}-\{s_j\})$ for some $0\leq i<j\leq n$. Put $S=\{x_{i+1},\dots,x_j\}$ and $m=\lfloor s_i\rfloor-\lfloor s_j\rfloor$;

then

$$\begin{aligned}\left|m+\sum_{s\in S}s\right| &= |m+s_j-s_i| \\ &= |\{s_j\}-\{s_i\}| \\ &\le \frac{1}{n+1},\end{aligned}$$

as desired. In case $1-t_n \le 1/(n+1)$, we take $S=\{x_1,\dots,x_n\}$ and $m=-\lceil s_n\rceil$, and again obtain the desired conclusion.

9.7. A set of 10 elements has $2^{10}-1=1023$ non-empty subsets. The possible sums of at most ten two-digit numbers cannot be larger than $10\cdot 99=990$. There are more subsets than possible sums, so two different subsets S_1 and S_2 must have the same sum. If $S_1\cap S_2=\emptyset$ then we are done. Otherwise remove the common elements and we get two non-intersecting subsets with the same sum.

9.8. Writing $y_i=\tan x_i$, with $-\frac{\pi}{2}\le x_i\le\frac{\pi}{2}$ ($i=1,\dots,7$), we have that

$$\frac{y_i-y_j}{1+y_iy_j}=\tan\left(x_i-x_j\right),$$

so all we need is to do is prove that there are x_i,x_j such that $0\le x_i-x_j\le\frac{\pi}{6}$. To do so we divide the interval $(-\frac{\pi}{2},\frac{\pi}{2})$ into 6 subintervals each of length $\frac{\pi}{6}$. By the box principle, two of the x_is will be in the same subinterval, and their difference will be not larger than $\frac{\pi}{6}$, as required.

9.9. Classify the numbers by their reminder when divided by 3. Either three of them will yield the same reminder, and their sum will be a multiple of 3, or there will be at least a number x_r for each possible reminder $r=0,1,2$, and their sum $x_0+x_1+x_2$ will be a multiple of 3 too.

9.10. We must prove that there are positive integers n,k such that

$$2009\cdot 10^k\le 2^n<2010\cdot 10^k\,.$$

That double inequality is equivalent to

$$\log_{10}(2009)+k\le n\log_{10}(2)<\log_{10}(2010)+k\,.$$

where $\log_{10}$ represents the decimal logarithm. Writing $\alpha=\log_{10}(2009)-3$, $\beta=\log_{10}(2010)-3$, we have $0<\alpha<\beta<1$, and the problem amounts to showing that for some integer n, the fractional part of $n\log_{10}(2)$ is in the interval $[\alpha,\beta)$. This is true because $\log_{10}(2)$ is irrational, and the integer multiples of an irrational number are dense modulo 1 (their fractional parts are dense in the interval $[0,1)$).

9.11. Let F be the face with the largest number m of edges. Then for the $m+1$ faces consisting of F and its m neighbors the possible number of edges are $3,4,\dots,m$. These are only $m-2$ possibilities, hence the number of edges must occur more than once.

10.1. After rationalizing we get a telescopic sum:

$$\begin{aligned}\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{99}+\sqrt{100}} &= (\sqrt{2}-1)+(\sqrt{3}-\sqrt{2})+\cdots+(\sqrt{100}-\sqrt{99}) \\ &= 10-1=9\,.\end{aligned}$$

10.2. We have

$$\sum_{n=1}^{N} n\cdot n! = \sum_{n=1}^{N}\{(n+1)-1\}\cdot n! = \sum_{n=1}^{N}\{(n+1)!-n!\} =$$
$$(2!-1!)+(3!-2!)+\cdots+((N+1)!-N!) = (N+1)!-1\,.$$

10.3. We have

$$\frac{6^k}{(3^{k+1}-2^{k+1})(3^k-2^k)} = \frac{3^k}{3^k-2^k}-\frac{3^{k+1}}{3^{k+1}-2^{k+1}}\,.$$

So this is a telescopic sum:

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{6^k}{(3^{k+1}-2^{k+1})(3^k-2^k)} &= \lim_{n\to\infty}\sum_{k=1}^{n}\left\{\frac{3^k}{3^k-2^k}-\frac{3^{k+1}}{3^{k+1}-2^{k+1}}\right\} \\ &= \lim_{n\to\infty}\left\{3-\frac{3^{n+1}}{3^{n+1}-2^{n+1}}\right\} \\ &= 3-1=2\,.\end{aligned}$$

10.4. This is a telescopic product:

$$\frac{n^3-1}{n^3+1} = \frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)} = \frac{(n-1)\{n\,(n+1)+1\}}{(n+1)\{(n-1)\,n+1\}}\,,$$

hence

$$\begin{aligned}\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1} &= \lim_{N\to\infty}\prod_{n=2}^{N}\frac{(n-1)\{n\,(n+1)+1\}}{(n+1)\{(n-1)\,n+1\}} \\ &= \lim_{N\to\infty}\frac{2\{N(N+1)+1\}}{3N(N+1)} = \frac{2}{3}\,.\end{aligned}$$

10.5. We have

$$\begin{aligned}\frac{n}{n^4+n^2+1} &= \frac{n}{(n^2+1)^2-n^2} \\ &= \frac{1/2}{n^2-n+1}-\frac{1/2}{n^2+n+1} \\ &= \frac{1/2}{(n-1)\,n+1}-\frac{1/2}{n\,(n+1)+1}\,.\end{aligned}$$

So

$$\sum_{n=0}^{N} \frac{n}{n^4+n^2+1} = \frac{1/2}{(-1)\cdot 0+1} - \frac{1/2}{0\cdot 1+1} + \frac{1/2}{0\cdot 1+1} - \frac{1/2}{1\cdot 2+1} + \cdots$$

$$\cdots + \frac{1/2}{(N-1)\,N+1} - \frac{1/2}{N\,(N+1)+1}$$

$$= \frac{1}{2} - \frac{1/2}{N\,(N+1)+1} \quad \underset{N\to\infty}{\longrightarrow} \quad \frac{1}{2}\,.$$

Hence, the sum is $1/2$.

11.1. Consider the function

$$f(x,y,z) = T(x,y,z) + T(y,z,x) + T(z,x,y) \quad = 4x^2 + 4y^2 + 4z^2\,.$$

On the surface of the planet that function is constant and equal to $4\cdot 20^2 = 1600$, and its average on the surface of the planet is 1600. Since the equation of a sphere with center in $(0,0,0)$ is invariant by rotation of coordinates, the three terms $T(x,y,z)$, $T(y,z,x)$, $T(z,x,y)$ have the same average value $\overline{T}$ on the surface of the planet, hence $1600 = 3\overline{T}$, and $\overline{T} = 1600/3$.

11.2. Writing $\alpha = \sqrt{2}$, the integrand $f(x) = 1/(1+\tan^\alpha x)$ verifies the following symmetry:

$$\begin{aligned} f(x) + f(\tfrac{\pi}{2} - x) &= \frac{1}{1+\tan^\alpha x} + \frac{1}{1+\cot^\alpha x} \\ &= \frac{1}{1+\tan^\alpha x} + \frac{\tan^\alpha x}{1+\tan^\alpha x} \\ &= 1\,. \end{aligned}$$

On the other hand, making the substitution $u = \frac{\pi}{2} - x$:

$$\int_0^{\frac{\pi}{2}} f(\tfrac{\pi}{2} - x)\,dx = -\int_{\frac{\pi}{2}}^{0} f(u)\,du = \int_0^{\frac{\pi}{2}} f(x)\,dx = I\,,$$

where I is the desired integral. So:

$$2I = \int_0^{\frac{\pi}{2}} \left\{f(x) + f(\tfrac{\pi}{2} - x)\right\}\,dx = \int_0^{\frac{\pi}{2}} 1\,dx = \frac{\pi}{2}\,.$$

Hence $I = \dfrac{\pi}{4}$.

11.3. The first player does have a winning strategy: place the first penny exactly on the center of the table, and then after the second player places a penny, place the next penny in a symmetric position respect to the center of the table. After each of the first player's move the configuration of pennies on the table will have radial symmetry, so if the second player can still place a penny somewhere on the table, the radially symmetric position respect to the center of the table will still not be occupied and the first player will also be able to place a penny there.

12.1. Let A be the set of positive integers no exceeding 1000 that are divisible by 7, and let B the set of positive integers not exceeding 1000 that are divisible by 11. Then $A \cup B$ is the set of positive integers not exceeding 1000 that are divisible by 7 or 11. The number of elements in A is $|A| = \left\lfloor \frac{1000}{7} \right\rfloor = 142$. The number of elements in B is $|B| = \left\lfloor \frac{1000}{11} \right\rfloor = 90$. The set of positive integers not exceeding 1000 that are divisible by 7 *and* 11 is $A \cap B$, and the number of elements in there is $|A \cap B| = \left\lfloor \frac{1000}{7 \cdot 11} \right\rfloor = 12$. Finally, by the inclusion-exclusion principle:

$$|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12 = 220 \,.$$

12.2. (This is equivalent to finding the number of onto functions from a n-element set to an k-element set.) If we remove the restriction "using all k flavors" then the first child can receive an ice-cream of any of the k available flavors, the same is true for the second child, and the third, etc. Hence the number of ways will be the product $k \cdot k \cdots k = k^n$.

Now we need to eliminate the distributions of ice-cream cones in which at least one of the flavors is unused. So let's call A_i = set of distributions of ice-creams in which at least the ith flavor is never used. We want to find the number of elements in the union of the A_i's (and later subtract it from k^n). According to the Principle of Inclusion-Exclusion that number is the sum of the elements in each of the A_i's, minus the sum of the elements of all possible intersections of two of the A_i's, plus the sum of the elements in all possible intersections of three of those sets, and so on. We have:

$|A_i| = (k-1)^n$ ($k-1$ flavors distributed among n children)

$|A_i \cap A_j| = (k-2)^n$ ($k-2$ flavors among n children)

$|$any triple intersection$| = (k-3)^n$ ($k-3$ flavors among n children)

and so on. On the other hand there are k sets A_i, $\binom{k}{2}$ intersections of two sets, $\binom{k}{3}$ intersections of three sets, etc. Hence the number of distributions of flavors that miss some flavor is

$$\binom{k}{1}(k-1)^n - \binom{k}{2}(k-2)^n + \binom{k}{3}(k-3)^n - \cdots \pm \binom{k}{k} 0^n \,,$$

and the number of distributions of flavors that do not miss any flavor is k^n minus the above sum, i.e.:

$$k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \binom{k}{3}(k-3)^n + \cdots \mp \binom{k}{k}0^n = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n\,.$$

12.3. Let's denote P_i the set of permutations fixing element a_i. The set of non-derangements are the elements of the union $P_1 \cup P_2 \cup \cdots \cup P_n$, and its number can be found using the inclusion-exclusion principle:

$$|P_1 \cup P_2 \cup \cdots \cup P_n| = \sum_i |P_i| - \sum_{i\neq j}|P_i \cap P_j| + \sum_{i\neq j\neq k\neq i}|P_i \cap P_j \cap P_k| - \cdots$$

Each term of that expression is the number of permutations fixing a certain number of elements. The number of permutations that fix m given elements is $(n-m)!$, and since there are $\binom{n}{m}$ ways of picking those m elements, the corresponding sum is $\binom{n}{m}(n-m)! = \frac{n!}{m!}$. Adding and subtracting from the total number of permutations $n!$, we get

$$D_n = n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n\frac{n!}{n!} = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!}\right).$$

13.1. - *First Solution:* The number of subsets of $\{1, 2, \ldots, n\}$ with odd cardinality is

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

The number of subsets of even cardinality is cardinality is

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$$

The difference is

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \pm \binom{n}{n} = (1-1)^n = 0\,.$$

- *Second Solution:* We define a bijection between the subsets with odd cardinality and those with even cardinality in the following way: if S is a subset with an odd number of elements we map it to $S' = S \cup \{1\}$ if $1 \notin S$, or $S' = S \setminus \{1\}$ if $1 \in S$.

13.2. (*Note:* see the section about recurrences for an alternate solution—here we use a combinatorial argument.)
We will prove that the number of k-element subsets of $\{1, 2, \ldots, n\}$ with no consecutive elements equals the number of all k-element subsets of $\{1, 2, \ldots, n-k+1\}$. To

do so we define a 1-to-1 correspondence between both kinds of subsets in the following way: to each subset $\{a_1, a_2, \dots, a_k\}$ $(a_1 < a_2 < \dots < a_k)$ of $\{1, 2, \dots, n\}$ without consecutive elements we assign the subset $\{a_1, a_2 - 1, \dots, a_i - i + 1, \dots, a_k - k + 1\}$ of $\{1, 2, \dots, n - k + 1\}$. We see that the mapping is in fact a bijection, with the inverse defined $\{b_1, b_2, \dots, b_i, \dots, b_k\} \mapsto \{b_1, b_2 + 1, \dots, b_i + i - 1, \dots, b_k + k - 1\}$. Hence, the number of k-element subsets of $\{1, 2, \dots, n\}$ with no consecutive elements is $\binom{n-k+1}{k}$. Note that the formula is valid also for $k = 0$ and $k = 1$.
Hence, the total number of subsets of $\{1, 2, \dots, n\}$ with no consecutive elements is the sum

$$\sum_{k=0}^{\lceil n/2 \rceil} \binom{n-k+1}{k}.$$

This sum is known to be equal to the shifted Fibonacci number F_{n+1}.

13.3. The probability of John getting n heads is the same as getting n tails. So the problem is equivalent to asking the probability of John getting as many tails as the number of heads gotten by Peter, and that is the same as both getting jointly a total of 20 heads. So the probability asked is the same as that of getting 20 heads after tossing $25 + 20 = 45$ coins, i.e.:

$$\binom{45}{20} 2^{-45}.$$

(That is $0.09009314767\dots$)

13.4. Let x be the distance from the man to the edge measured in steps. For $n > 0$, let P_n the probability that the drunken man ends up over the edge when he starts at $x = n$ steps from the cliff. Then $P_1 = (1 - p) + pP_2$. We now rewrite P_2 in the following way. Paths from $x = 2$ to $x = 0$ can be broken into two parts: a path that goes from $x = 2$ to $x = 1$ for the first time, and a path that goes from $x = 1$ to $x = 0$. The probability of the latter is P_1, because the situation is exactly the same as at the beginning. The probability of the former is also P_1, because the structure of problem is identical to the original one with x increased by 1. Since both probabilities are independent, we have $P_2 = P_1^2$. Hence

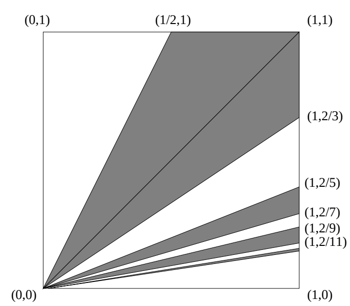
$$P_1 = (1 - p) + pP_1^2.$$

Solving this equation we get two solutions, namely $P_1 = 1$ and $P_1 = \dfrac{1-p}{p}$.
We now need to determine which solution goes with each value of p. For $p = 1/2$ both solutions agree, and then $P_1 = 1$. For $p = 0$ we have $P_1 = 1$, and when $p = 1$, $P_1 = 0$, because the man always walks away from the cliff. For $0 < p < 1/2$ the second solution is impossible, so we must have $P_1 = 1$. For $1/2 < p \leq 1$ we have that the second solution is strictly less than 1. By continuity P_1 cannot take both values 1 and $\dfrac{1-p}{p}$ on the interval $(1/2, 1]$, so since $P_1 = 0$ for $p = 1$, we must have

$P_1 = \dfrac{1-p}{p}$ on that interval. Hence, the probability of escaping the cliff is

$$1 - P_1 = \begin{cases} 0 & \text{if } 0 \le p \le \frac{1}{2}\,, \\ 2 - \dfrac{1}{p} & \text{if } \frac{1}{2} < p \le 1\,. \end{cases}$$

13.5. The set $\{(X,Y) \mid X,Y \in (0,1)\}$ is the unit square square with corners in $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$, whose area is 1. The desired probability will be the area of the subset of points (X,Y) in that square such that the closest integer to X/Y is odd. The condition "the closest integer to X/Y is odd" is equivalent to $|X/Y - (2n+1)| < 1/2$ for some non-negative integer n, or equivalently, $2n + 1/2 < X/Y < 2n + 3/2$. That set of points is the space in the unit square between the lines $Y = \frac{2}{4n+1}X$ and $Y = \frac{2}{4n+3}X$. That area can be decomposed into triangles and computed geometrically (see figure.)



For $n = 0$ the area is $1/4 + 1/3$. For $n \ge 1$ it is $\frac{1}{4n+1} - \frac{1}{4n+3}$. Hence the total area is

$$P = \frac{1}{4} + \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

We can find the sum of that series using the Gregory-Leibniz series:

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}\,,$$

and we get

$$P = -\frac{1}{4} + \frac{\pi}{4} = 0.5353981635\ldots\,.$$

13.6. Let L_1, L_2, and L_3 the lengths of those three arcs. We have $L_1 + L_2 + L_3 = 2\pi$. On the other hand the expected value of several random variables is additive:

$$E[L_1 + L_2 + L_3] = E[L_1] + E[L_2] + E[L_3]\,.$$

By symmetry $E[L_1] = E[L_2] = E[L_3]$, and the sum must be 2π, hence each expected value is $\frac{2\pi}{3}$. So, this is the answer, the expected value of the arc containing the point $(1,0)$ is $\frac{2\pi}{3}$.

13.7. The problem is equivalent to dropping two random points on an interval of length 9 inches. By identifying the two endpoints of the interval the problem becomes identical to dividing a circle of length 9 at three points chosen at random. The expected values of their lengths must add to 9 inches, and by symmetry they should be the same, so each expected value must be 3 inches. Hence, this is the answer, the average length of the fragment with the blue dot will be 3 inches.

13.8. Label the points $x_0, x_1, x_2, \dots, x_n$ ($x_n \equiv x_0$.) Then the center of the circle will *not* be in the polygon if and only if one of the arcs defined by two consecutive points (measured counterclockwise) is greater than π. Let E_k ($k = 0, \dots, n-1$) be the event "the arc from x_k to the point next to x_k (counterclockwise) is larger than π." The probability of each E_k is obviously $\frac{1}{2^{n-1}}$, because for it to happen all points other than x_k must lie in the same half-circle ending at x_k. On the other hand, the events $E_0, E_1, \dots, E_{n-1}$ are incompatible, i.e., no two of them can happen at the same time. Then, the probability of one of them happening is the sum of the probabilities:

$$P(E_0 \text{ or } E_1 \text{ or } \cdots \text{ or } E_{n-1}) = P(E_0) + P(E_1) + \cdots + P(E_{n-1}) = \frac{n}{2^{n-1}} .$$

Hence, the desired probability is $1 - \dfrac{n}{2^{n-1}}$.

14.1. Let $I = \dfrac{10^{20000} - 3^{200}}{10^{100} + 3} = \dfrac{(10^{100})^{200} - 3^{200}}{10^{100} + 3}$
$$= (10^{100})^{199} - (10^{100})^{198} \cdot 3 + \cdots + 10^{100} \cdot 3^{198} - 3^{199}$$
so I is an integer. On the other hand since $\frac{3^{200}}{10^{100}+3} < 1$ we have that $\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor = I$. Finally the rightmost digit of I can be found as the 1-digit number congruent to $-3^{199} \pmod{10}$. The sequence $3^n \bmod 10$ has period 4 and $199 = 3 + 4 \cdot 49$, hence $-3^{199} \bmod 10 = -3^3 \bmod 10 = -27 \bmod 10 = 3$. Hence the units digit of I is 3.

14.2. Let α be any (say the smallest) acute angle of a right triangle with sides 3, 4 and 5 (or any other Pythagorean triple). Next, place an infinite sequence of points on the unit circle at coordinates $(\cos(2n\alpha), \sin(2n\alpha))$, $n = 0, 1, 2, \dots$ (The sequence contains in fact infinitely many points because α cannot be a rational multiple of π.) The distance from $(\cos(2n\alpha), \sin(2n\alpha))$ to $(\cos(2m\alpha), \sin(2m\alpha))$ is $2\sin(|n - m|\alpha)$, so all we need to prove is that $\sin(k\alpha)$ is rational for any k. This can be done by induction using that $\sin\alpha$ and $\cos\alpha$ are rational, and if $\sin u$, $\cos u$, $\sin v$ and $\cos(v)$ are all rational so are $\sin(u + v) = \sin u \cos v + \cos u \sin v$ and $\cos(u + v) = \cos u \cos v - \sin u \sin v$.

14.3. We can prove the first part by way of contradiction. Assume that we have colored the points of the plane with three colors such that any two points at distance 1 have different colors. Consider any two points A and B at distance $\sqrt{3}$ (see figure 3). The circles of radius 1 and center A and B meet at two points P and Q, forming equilateral triangles APQ and BPQ. Since the vertices of each triangle must have different colors that forces A and B to have the same color. So any two points at distance $\sqrt{3}$ have the same color. Next consider a triangle DCE with

$CD = CE = \sqrt{3}$ and $DE = 1$. The points D and E must have the same color as C, but since they are at distance 1 they should have different colors, so we get a contradiction.

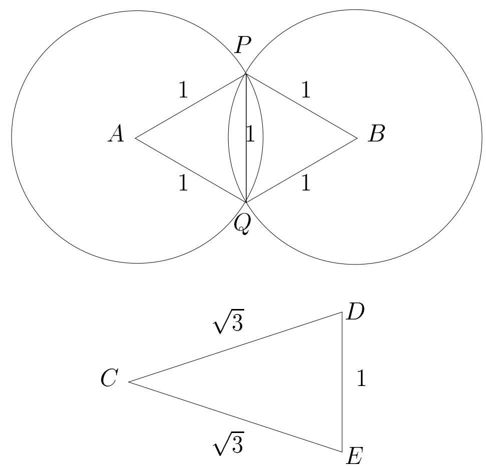


FIGURE 3

For the second part, if we replace "three" by "nine" then we can color the plane with nine different colors so that any two points at distance 1 have different colors: we can arrange them periodically in a grid of squares of size $2/3 \times 2/3$ as shown in figure 4. If two points P and Q have the same color then either they belong to the same square and $PQ < (2/3)\sqrt{2} < 1$, or they belong to different squares and $PQ \geq 4/3 > 1$.

14.4. Since the values are positive integers, one of them, say n, will be the smallest one. Look at any square with that value n. Since the values of its four neighbors must be at least n and their average is n, all four will have value n. By the same reasoning the neighbors of these must be n too, and so on, so all the squares must have the same value n.

14.5. One or two points are obviously insufficient, but three can do it. Choose $\alpha \in \mathbb{R}$ so that α^2 is irrational, for example $\alpha = \sqrt[3]{2}$. Use punches at $A = (-\alpha, 0)$, $B = (0, 0)$, and $C = (\alpha, 0)$. If $P = (x, y)$ then

$$AP^2 - 2BP^2 + CP^2 = (x+\alpha)^2 + y^2 - 2(x^2+y^2) + (x-\alpha)^2 + y^2 = 2\alpha^2$$

is irrational, so AP, BP, CP cannot all be rational.

14.6. We have

$$f(n+2) - f(n+1) = (n+2)! = (n+2)(n+1)! = (n+2)(f(n+1) - f(n)),$$

A	B	C	A	B	C
D	E	F	D	E	F
G	H	I	G	H	I
A	B	C	A	B	C
D	E	F	D	E	F

FIGURE 4

hence

$$\begin{aligned} f(n+2) &= (n+2)(f(n+1)-f(n)) + f(n+1) \\ &= (n+3)f(n+1) - (n+2)f(n)\,, \end{aligned}$$

and we can take $P(x) = x+3$, $Q(x) = -x-2$.

14.7. A conspiratorial subset of $S = \{1,2,\dots,16\}$ has at most two elements from $T = \{1,2,3,5,7,11,13\}$, so it has at most $2+16-7 = 11$ numbers. On the other hand all elements of $S \setminus T = \{4,6,8,9,10,12,14,15,16\}$ are multiple of either 2 or 3, so adding 2 and 3 we obtain the following 11-element conspiratorial subset:

$$\{2,3,4,6,8,9,10,12,14,15,16\}\,.$$

Hence the answer is 11.

14.8. The statement is true. Let ϕ any bijection on F with no fixed points ($\phi(x) \neq x$ for every x), and set $x * y = \phi(x)$. Then

(i) $x*z = y*z$ means $\phi(x) = \phi(y)$, and this implies $x = y$ because ϕ is a bijection.

(ii) We have $x*(y*z) = \phi(x)$ and $(x*y)*z = \phi(\phi(x))$, which cannot be equal because that would imply than $\phi(x)$ is a fixed point of ϕ.

14.9. For a given partition π of $\{1,2,3,4,5,6,7,8,9\}$, no more than three different values of $\pi(x)$ are possible (four would require one part each of size at least 1,2,3,4, and that's already more than 9 elements). If no such x, y exist, each pair $(\pi(x), \pi'(x))$ occurs for at most 1 element of x, and since there are only 3×3 possible pairs, each must occur exactly once. In particular, each value of $\pi(x)$ must occur 3 times. However, clearly any given value of $\pi(x)$ occurs $k\pi(x)$ times, where k is the number of distinct parts of that size. Thus $\pi(x)$ can occur 3 times only if it equals 1 or 3, but we have three distinct values for which it occurs, contradiction.

14.10. The answer is $2n-3$.
Note that this number is attained with $S = \{1, 2, 3, \ldots, n\}$ because

$$2A_S = \{3, 4, \ldots, n, n+1, n+2, \ldots, 2n-1\}$$

has cardinality $2n-3$. It remains to prove that $2n-3$ is in fact minimum. That is so because for any $A = \{a_1, a_2, \ldots, a_n\}$ with $a_1 < a_2 < \cdots < a_n$ we have that

$$\underbrace{a_1 + a_2 < a_1 + a_3 < a_1 + a_4 < \cdots < a_1 + a_n}_{n-1} < \underbrace{a_2 + a_n < \cdots < a_{n-1} + a_n}_{n-2}$$

are $2n-3$ distinct numbers.

14.11. We can group the terms of the sequence in the following way:

$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1}_{b_0} + \underbrace{(a_2 + a_3)}_{b_1} + \underbrace{(a_4 + a_5 + a_6 + a_7)}_{b_2} + \cdots + \underbrace{(a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1})}_{b_k} + \cdots$$

The condition implies that $b_k \leq b_{k+1}$ for every $k \geq 0$, hence the sequence diverges.

14.12. Alice adds the values of the coins in odd positions 1st, 3rd, 5th, etc., getting a sum S_{odd}. Then she does the same with the coins placed in even positions 2nd, 4th, 6th, etc., and gets a sum S_{even}. Assume that $S_{odd} \geq S_{even}$. Then she will pick all the coins in odd positions, forcing Bob to pick only coins in the even positions. To do so she stars by picking the coin in position 1, so Bob can pick only the coins in position 2 or 50. If he picks the coin in position 2, Alice will the pick coin in position 3, if he picks the coin in position 50 she picks the coin in position 49, and so on, with Alice always picking the coin at the same side as the coin picked by Bob.
If $S_{odd} \leq S_{even}$, then Alice will use a similar strategy ensuring that she will end up picking all the coins in the even positions, and forcing Bob to pick the coins in the odd positions—this time she will pick first the 50th coin, and then at each step she will pick a coin at the same side as the coin picked by Bob.

14.13. By contradiction. If the equality $f(x) = x$ never holds then $f(x) > x$ for every x, or $f(x) < x$ for every x. Then $f(f(x)) > f(x) > x$ for every x, or $f(f(x)) < f(x) < x$ for every x, contradicting the hypothesis that $f \circ f$ has a fixed point.

14.14. By contradiction. Assume $\tan 1°$ is rational. Then $\tan 2° = \tan(1° + 1°) = \frac{\tan 1° + \tan 1°}{1 - \tan 1° \tan 1°}$ would be rational too. Same for $\tan 3° = \tan(2° + 1°) = \frac{\tan 2° + \tan 1°}{1 - \tan 2° \tan 1°}$, ..., $\tan(n+1)° = \tan(n° + 1°) = \frac{\tan n° + \tan 1°}{1 - \tan n° \tan 1°}$, ..., $\tan 30° = \tan(29° + 1°) = \frac{\tan 29° + \tan 1°}{1 - \tan 29° \tan 1°}$. But $\tan 30° = \frac{1}{\sqrt{3}}$ is irrational.

14.15. Let $x_1 = 5\sqrt{5} + 11$, $x_2 = -5\sqrt{5} + 11$. These numbers are roots of the polynomial $(x - x_1)(x - x_2) = x^2 - 22x - 4$, and consequently the sequence $a_n = x_1^n + x_2^n$ verifies the recurrence:

$$a_{n+2} = 22a_{n+1} + 4a_n \,.$$

We also have $a_0 = 2$, $a_1 = x_1 + x_2 = 22$, hence by induction we get that a_n is even for every $n \geq 0$. On the other hand $|x_2| = 5\sqrt{5} - 11 = \frac{4}{5\sqrt{5}+11} < \frac{2}{11} < 1$, hence $0 < |x_2|^n < 1$. Since $x_2 < 0$ we have that $x_1^{2n+1} = a_{2n+1} - x_2^{2n+1} = a_{2n+1} + |x_2|^{2n+1}$, where a_{2n+1} is even, and $0 < |x_2|^{2n+1} < 1$. So the integer part of x_1^{2n+1} equals a_{2n+1}, which is even, Q.E.D.