## 1. HOMOTOPY, SUSPENSION AND LOOP

We (tend to) only care things up to homotopy. But what is homotopy?

**Definition 1.1.** Given maps $f$ and $g : X \to Y$, a **homotopy** $h$ between $f$ and $g$ is a map $h : X \times I \to Y$ ($I = [0,1]$) such that $f(x) = h(x,0)$ and $g(x) = h(x,1)$. We say $f$ and $g$ are **homotopic** if there exists a homotopy between them.

This is the defination of homotopy between maps of spaces, the concept can be generalized to much general cases. The idea is we are doing $X$ cross an interval, or an analog of the notion of intervals (so up the dimension by 1).

Homotopy is an equivalent relation, the equivalent class of $X$ under this relation is called the homotopy class of $X$. The homotopy classes of the (based) maps from $S^1$ (the circle) to a space $X$ form a group, the first homotopy group of $X$, denote $\pi_1(X)$. Usually a space's homotopy type can be detected by its homotopy groups.

**Definition 1.2.** Given a space $X$, the **cone** on $X$ is the quotient space $X \times I/\sim$, where $(x,1) \sim (x',1)$.

**Definition 1.3.** Given a map $f : X \to Y$, the **mapping cone** $C_f$ is defined to be the quotient space $X \times I \sqcup_f Y/\sim$ with respect to the equivalence relation $(x,0) \sim (x',0)$, $(x,1) \sim f(x)$.

A cone is contractible, so roughly speaking the mapping cone is missing the homotopy information of $f(X)$. In a special case where $X$ is a subspace of $Y$, the mapping cone only see the homotopy information of $Y \backslash X$.

What if we make a double-sided cone? It might no longer be contractible! The **suspension** of $X$ can be described as the quotient space $X \times I/\sim$, where $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$. But we also have another way to write out the definition.

**Exercise 1.4.** *Show that* $\Sigma S^1 \cong S^2$*, deduct* $\Sigma S^n \cong S^{n+1}$.

Let $X$ and $Y$ be based spaces, the smash product of $X$ and $Y$ is defined as the quotient $X \times Y/X \vee Y$, where $X \vee Y$ is the one point union of $X$ and $Y$, denote $X \wedge Y$.

**Exercise 1.5.** *Show that* $S^1 \wedge S^1 \cong S^2$*, deduct* $S^1 \wedge S^n \cong S^{n+1}$.

**Definition 1.6.** Let $X$ be a based space. The **suspension** of $X$ is defined as $S^1 \wedge X$, the notation is $\Sigma X$. The **loop space** of $X$ is the homotopy classes of the based maps $[S^1, X]_*$ equipped with compact open topology, denote $\Omega X$.

Note that $\Omega X$ and $\pi_1(X)$ has the same underlying set, if $X$ is path connected. We will see later why suspention and loop play an essential rule.

## 2. SYMMETRIC MONOIDAL CATEGORY, TENSOR HOM ADJUNCTION

Now we come to the most abstract part of math, where we only look at the essense of everything and try to live in this.

**Definition 2.1.** A **category** $\mathscr{C}$ consists of

- a collection of **objects**;
- a collection of **arrows** between objects;
- a way to **compose** arrows.

Note that for every objects, there is an identity arrow between it and itself.

**Definition 2.2.** A **functor** $F$ from a category $\mathscr{C}$ to a category $\mathscr{D}$ is a map sending each object $c \in \mathscr{C}$ to an object $F(c) \in \mathscr{D}$, and each arrow $f : c \to c'$ to an arrow $F(f) : F(c) \to F(c')$ (or contravariant case, $F(f) : F(c) \leftarrow F(c')$) in $\mathscr{D}$, such that

- $F$ preserves identity arrows;
- $F$ preserves compositions.

**Example 2.3.**

(1) **Vect**-vector spaces as objects, linear maps as arrows;
(2) **Gp**-groups as objects, group homomorphisms as arrows;
(3) **AbGp**-abelian groups as objects, group homomorphisms as arrows;
(4) **Top**-topological spaces as objects, continuous maps as arrows;
(5) $\mathbf{Mod}_R$-for a commutative ring $R$, $R$-modules as objects, $R$-module maps as arrows.

**Example 2.4.**

(1) the (first) homotopy group $\pi_1$ can be considered as a functor from $\mathbf{Top}_*$ to **Gp**;
(2) given a ring map $R \to S$, it is defining a functor from $\mathbf{Mod}_R$ to $\mathbf{Mod}_S$;
(3) suspension and loop can both be considered as functors from $\mathbf{Top}_*$ to itself.

**Exercise 2.5.** *Verify the suspension and loop are functors.*

Not in all categories we can multiply things, yet in those we care a lot, we do.

**Definition 2.6.** A symmetric monoidal category is a category $(\mathcal{C}, \otimes, I)$, with bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an object $I \in \mathcal{C}$ that is called the unit, three natural isomorphisms $\alpha_a : a \otimes I \cong a$, $\beta_{a,b} : a \otimes b \cong b \otimes a$, $\gamma_{a,b,c} : a \otimes (b \otimes c) \cong (a \otimes b) \otimes c$ such that the following is true / diagrams commute:

(1) Commutativity: $\beta_{a,b} \circ \beta_{b,a} = \mathrm{id}$;
(2) Unital (also known as the triangle identity):

$$\begin{array}{ccc} a \otimes (I \otimes c) & \xrightarrow{\gamma} & (a \otimes I) \otimes c \\ \downarrow{\scriptstyle 1\otimes\beta} & & \downarrow{\scriptstyle \alpha\otimes 1} \\ a \otimes (c \otimes I) & \xrightarrow{1\otimes\alpha} & a \otimes c \end{array} ;$$

(3) The associativity pentagon: the vertices are the five possible orders to multiply four elements in binary ways and the edges are one-step associativities;
(4) "$C_3$-equivariant" (also known as the hexagon identity):

$$\begin{array}{ccccc} a \otimes (b \otimes c) & \xrightarrow{\gamma} & (a \otimes b) \otimes c & \xrightarrow{\beta} & c \otimes (a \otimes b) \\ \downarrow{\scriptstyle 1\otimes\beta} & & & & \downarrow{\scriptstyle \gamma} \\ a \otimes (c \otimes b) & \xrightarrow{\gamma} & (a \otimes c) \otimes b & \xrightarrow{\beta\otimes 1} & (c \otimes a) \otimes b \end{array} .$$

**Definition 2.7.** Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. A monoid in $\mathcal{C}$ is an object $M \in \mathcal{C}$ equipped with unit and multiplication maps

$$\eta : I \to M \text{ and } \mu : M \otimes M \to M,$$

such that the following diagrams are commutative:

$$
\begin{tikzcd}
I \otimes M \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\alpha"'] & M \otimes M \arrow[d, "\mu"] & M \otimes I \arrow[l, "\mathrm{id}\otimes\eta"'] \arrow[dl, "\alpha"] \\
& M &
\end{tikzcd},
$$

$$
\begin{tikzcd}
(M \otimes M) \otimes M \arrow[rr, "\gamma"] \arrow[d, "\mu\otimes\mathrm{id}"'] & & M \otimes (M \otimes M) \arrow[d, "\mathrm{id}\otimes\mu"] \\
M \otimes M \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\
& M &
\end{tikzcd}.
$$

A commutative monoid in $\mathcal{C}$ is a monoid $M$ satisfying the addition diagram:

$$
\begin{tikzcd}
M \otimes M \arrow[rr, "\beta"] \arrow[dr, "\mu"'] & & M \otimes M \arrow[dl, "\mu"] \\
& M &
\end{tikzcd}
$$

**Example 2.8.** A (commutative) monoid in the category of set is a (commutative) monoid. Here, the second "monoid" means a group without inverse.

**Remark 2.9.** Terminology: Sometimes people use the words "monoid" and "algebra" are used interchangeably to mean the same thing in a symmetric monoidal category. The reason is probably because a commutative monoid is the same thing as an algebra over the commutative opeard. They only slightly differ in that "algebra" is used for a general symmetric monoidal category while "monoid" is used for a Cartesian monoidal category.

However, we point out that an algebra in algebra has two binary operations while an algebra in topology has only one.

We can consider this "product" as another kind of functor, defined on a product category.

**Example 2.10.** Let $R$ be a commutative ring. Consider $- \otimes_R - : \mathbf{Mod}_R \times \mathbf{Mod}_R \to \mathbf{Mod}_R$ sending $(M, N)$ to $M \otimes_R N$, this is a functor. And $(\mathbf{Mod}_R, \otimes_R, R)$ is a symmetric monoidal category.

**Example 2.11.**

(1) $(\mathbf{Top}, \times, *)$
(2) $(\mathbf{Top}_*, \wedge, S^0)$;
(3) $(\mathbf{Vect}, \oplus, k)$;
(4) $(\mathbf{AbGp}, \oplus, 0)$;
(5) $(\mathbf{AbGp}, \otimes_{\mathbb{Z}}, \mathbb{Z})$.

Two functors can be inverse of each other, but that is way too strong. Let $\mathrm{Hom}_{\mathscr{C}}(-,-)$ be a functor from $\mathscr{C} \times \mathscr{C}$ to **Set**, the resulting set is all the arrows from a selected object to another in $\mathscr{C}$. The target of this functor doesn't have to be in **Set**.

**Example 2.12.** $\mathrm{Hom}_{\mathbf{Top}_*}(X, Y)$ is a based space equipped with the compact open topology, i.e. $\mathrm{Hom}_{\mathbf{Top}_*}(-,-)$ is a functor from $\mathbf{Top}_* \times \mathbf{Top}_*$ to $\mathbf{Top}_*$.

Similarly, for a commutative ring $R$, one can verify that $\mathrm{Hom}_{\mathbf{Mod}_R}(M,N)$ can be given a $R$-module structure.

Assuming we have a functor $L : \mathscr{C} \to \mathscr{D}$ and another functor $R : \mathscr{D} \to \mathscr{C}$ between two categories $\mathscr{C}$ and $\mathscr{D}$. We can compare $\mathrm{Hom}_{\mathscr{C}}(c, R(d)$ and $\mathrm{Hom}_{\mathscr{D}}(L(c), d)$ for any $c \in \mathscr{C}$ and $d \in \mathscr{D}$. This may be thought of as a generalized version of two functors behave like inverses to each other.

**Definition 2.13.** Inheriting the setting above. $L$ and $R$ are called **a pair of adjoint functors** (or often an **adjunction**) if there is a bijection

$$\mathrm{Hom}_{\mathscr{C}}(c, R(d) \cong \mathrm{Hom}_{\mathscr{D}}(L(c), d)$$

for each $c \in \mathscr{C}$ and $d \in \mathscr{D}$, and for any arrows $c \to c'$, there is a square

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(c, R(d)) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\mathscr{D}}(L(c), d) \arrow[d] \\ \mathrm{Hom}_{\mathscr{C}}(c', R(d)) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathscr{D}}(L(c'), d) \end{tikzcd}$$

for any arrows $d \to d'$, there is a square

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{C}}(c, R(d)) \arrow[r, "\cong"] \arrow[d] & \mathrm{Hom}_{\mathscr{D}}(L(c), d) \arrow[d] \\ \mathrm{Hom}_{\mathscr{C}}(c, R(d')) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathscr{D}}(L(c), d') \end{tikzcd}$$

Those square conditions are often denoted **natural in** $c$ **and** $d$.

**Example 2.14.** This is the celebrated **tensor-hom adjunction**: Let $N$ be a $(R,S)$-bimodule, then $- \times_R N$ and $\mathrm{Hom}_S(N, -)$ is a pair of adjoint functors. i.e.

$$\mathrm{Hom}_S(M \otimes_R N, K) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(N, K))$$

for any $R$-module $M$ and $S$-module $K$.

**Example 2.15.** Given $- \wedge X$ and $\mathrm{Map}_*(X, -)$ is a pair of adjunction, deduct that suspension $\Sigma$ and loop $\Omega$ is a pair of adjunction on the homotopy classes, i.e.

$$[\Sigma Y, Z]_* \cong [Y, \Omega Z]_*$$

This adjunction plays an essential rule in the fundation of spectra.

## 3. THE EILENBERG-STEENROD AXIOMS, COHOMOLOGY THEORIES AND SPECTRA

Some of us are familiar with the ordinary cohomology groups, others are not. If we squeeze the essense out of those, i.e. axiomize the good and essential properties they satisfy, we get a very abstract family of things.

**Definition 3.1.** Let $\mathbf{Top}_*$ be the category of pointed spaces and pointed maps. A **reduced cohomology theory** $h^*$ is a collection of contravariant functors and natural equivalences indexed by integers

$$h^n : \mathbf{Top}_* \to \mathbf{AbGp}$$
$$s^n : h^n \circ \Sigma \to h^{n-1}$$

satisfying:

- Homotopy: If $f_0 \simeq f_1 : X \to Y$, then $f_0^* = f_1^* : h^n(Y) \to h^n(X)$ for all $n \in \mathbb{Z}$.
- Exactness: For every $A \subseteq X$, we have exact sequence

$$h^n(X \cup CA) \xrightarrow{j^*} h^n(X) \xrightarrow{i^*} h^n(A)$$

where $i : A \overset{i}{\hookrightarrow} X$ is the inclusion and $j : X \overset{j}{\hookrightarrow} X \cup CA$ is the canonical inclusion into the cone of $i$.

**Remark 3.2.** Exact means the kernel of the map is the same as the image of the previous one, in the language above, it is interprated as $ker(i^*) = im(j^*)$.

The cohomology groups of a space is a cohomology theory, and there are more. We can transport the data to a sequal of spaces with links between them via the Brown representibility theorem, which is literally saying for every $n \in \mathbb{Z}$ there exists a space $C_n$ such that $h^n(X) \cong [X, C_n]_*$ natural in $X$. Thus we can look at $C_n$s instead of $h^*$.

**Definition 3.3.** A **spectrum** $X$ is a sequence of pointed spaces $X_n$ with structure maps $\epsilon_n : \Sigma X_n \to X_{n+1}$. A function $f : X \to Y$ of degree $r$ between two spectra is a sequence of maps $f_n : X_n \to Y_{n-r}$ that is strictly compatible with the structure maps.

The structure maps can be given in terms of suspension: $\Sigma X_n \to X_{n+1}$, or in terms of loop: $X_n \to \Omega X_{n+1}$, via the bridge of $\Sigma - \Omega$ adjunction. the natural equivalences incoded in the axiom of cohomology theories suggest us, even though a spectrum contains a sequence of spaces, but essentially, up to homotopy, we should think them as one space.