FAMILIES OF SPACE CURVES WITH LARGE COHOMOLOGY

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Abstract. We investigate space curves with large cohomology. To this end we introduce curves of subextremal type. This class includes all subextremal curves. Based on geometric and numerical characterizations of curves of subextremal type, we show that, if the cohomology is "not too small," then they can be parameterized by the union of two generically smooth irreducible families; one of them corresponds to the subextremal curves. For curves of negative genus, the general curve of each of these families is also a smooth point of the support of an irreducible component of the Hilbert scheme. The two components have the same (large) dimension and meet in a subscheme of codimension one.

Contents

1. Introduction

In this note we study space curves of degree d and (arithmetic) genus g that have large cohomology. Since in this case the cohomology puts only little restrictions on the curves to deform, one expects that such curves form large families. In fact, Martin-Deschamps and Perrin have shown that among the curves C with fixed d and g, there are curves that maximize the Rao function $h^1(\mathcal{I}_C(j))$ for all $j \in \mathbb{Z}$. Such curves are called extremal curves. They also showed in [12] that the extremal curves form a family whose closure in the Hilbert scheme $H_{d,g}$ of locally Cohen-Macaulay curves is, topologically, a generically smooth component of $H_{d,g}$.

If one excludes the extremal curves, Nollet [15] showed that among the remaining curves, there are again curves that maximize $h^1(\mathcal{I}_C(j))$ for all $j \in \mathbb{Z}$. These curves are called subextremal. However, one cannot continue in this fashion. Among the curves that are neither extremal nor subextremal, there is no curve that maximizes the Rao functions in *all* degrees. This motivates our definition of curves of subextremal type. These are curves that have the same Rao function as the subextremal curves in all degrees $j = 1, \ldots, d-3$. Each such curve is contained in a unique quadric that is either reducible or not reduced.

It turns out that the curves of subextremal type can be parameterized by two irreducible and generically smooth families that have the same dimension. One of them corresponds to subextremal curves. The curves in the other family have the property

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that each of them is contained in a quadric that is not reduced, i.e. in a double plane. Furthermore, we show that if $g < 0$, then the closure of each of the two families in $H_{d,g}$ is, topologically, a generically smooth irreducible component of $H_{d,g}$. The two components meet in a subscheme of codimension one that corresponds to the subextremal curves that are contained in a double plane.

The paper is organized as follows:

Section 2 contains some preliminary results. After recalling the definitions and characterizations of subextremal and extremal curves, we establish some useful tools. We discuss the residual sequence of a curve C with respect to a hyperplane H that contains a planar subcurve $C' \subset C$. This sequence determines a zero-dimensional subscheme $Z \subset H$. For curves of subextremal type, Z turns out to be contained in a conic. This puts heavy restrictions on Z which are pointed out at the end of Section 2.

The following section is devoted to the structure of curves of subextremal type (Theorem 3.2). In particular, we show that a curve of subextremal type can be characterized by the values of its Hilbert function in degree two and three. Geometrically, it is distinguished by the fact that it contains a planar subcurve of degree $d-2$ and that the residual curve C' is a planar conic. This is used to completely describe the Rao functions of curves of subextremal type (Theorem 3.5). In addition to the degree d and genus g, each such Rao function is determined by an integer b where b can take only finitely many values.

In Section 4, we investigate numerical invariants of a curve C of subextremal type. Whereas the postulation character of C depends only on its degree and genus, its graded Betti numbers are determined by the triple (d, g, b) and depend also on b. This is a consequence of results in [5] which we also apply to determine the defining equations of the curves that are not subextremal.

In Section 5 we begin our study of families of curves of subextremal type. More precisely, we exhibit two families which are distinct if the cohomology is not too small. The first one parameterizes the curves of subextremal type that are contained in a double plane. This family can be stratified according to the Rao function of its curves. The curves with minimal Rao function form an open and generically smooth subfamily. The second family is formed by the subextremal curves. Both families give rise to irreducible and generically smooth subschemes of the Hilbert scheme $H_{d,g}$. They have the same (large) dimension, which is explicitly computed.

In Section 6 we give a geometric description of the general curve in each of the two families. We use it to show that the closure of the two subschemes of $H_{d,g}$ corresponding to the two families of curves of subextremal type are actually the support of irreducible components of $H_{d,g}$, provided $g < 0$. Thus, in this case the Hilbert scheme $H_{d,g}$ contains besides the component that parameterizes extremal curves two further components, one is smooth at the general subextremal curve, the other is smooth at the general curve of subextremal type that is contained in a double plane.

2. Preliminary results and background

We collect here some results that will be used later on.

2.1. Standing Notation

- K: algebraically closed field of characteristic zero.
- $\mathbb{P}^n$ the n-dimensional projective space over K.
- For a closed subscheme $X \subseteq \mathbb{P}^n$, h_X denotes the Hilbert function of X and ∂h_X denotes the first difference of h_X, i.e. $\partial h_X(j) = h_X(j) - h_X(j-1)$.

- If $X \subseteq \mathbb{P}^n$ is a closed subscheme, then $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbb{P}^n}$ denotes the ideal sheaf of X and $I_X \subseteq K[X_0, \dots, X_n]$ denotes the (saturated) homogeneous ideal of X.
- We agree that the empty subscheme of $\mathbb{P}^n$ has degree 0.
- $C \subseteq \mathbb{P}^3$: non-degenerate, projective curve of degree d and arithmetic genus g, where curve means a pure 1-dimensional projective subscheme (i.e. without 0-dimensional components); in particular C is locally Cohen-Macaulay.
- Γ: general hyperplane section of C.
- If C is a curve, the function $\rho_C(j) := h^1(\mathcal{I}_C(j))$ ($j \in \mathbb{Z}$) is called the *Rao function of C*.

2.2. Extremal and subextremal curves

Now we recall some results on curves having large cohomology, which were one of the starting points for our investigation. These curves were studied by Martin-Deschamps and Perrin [11], Ellia [6], and Nollet [15].

Martin-Deschamps and Perrin in [11] proved that for $d \geq 2$ the Rao function of C satisfies the inequality $\rho_C(j) \leq \rho^E(j)$, where $\rho^E : \mathbb{Z} \to \mathbb{Z}$ is the function defined by:

$$\rho^E(j) := \begin{cases} 0 & \text{if } \; j \leq -\binom{d-2}{2} + g \\ \binom{d-2}{2} - g + j & \text{if } \; -\binom{d-2}{2} + g \leq j \leq 0 \\ \binom{d-2}{2} - g & \text{if } \; 0 \leq j \leq d-2 \\ \binom{d-1}{2} - g - j & \text{if } \; d-2 \leq j \leq \binom{d-1}{2} - g \\ 0 & \text{if } \; \binom{d-1}{2} - g \leq j. \end{cases}$$

A non-degenerate curve $C \subseteq \mathbb{P}^3$ such that $\rho_C(j) = \rho^E(j)$ for every $j \in \mathbb{Z}$, is called *extremal* (see [11]).

For extremal curves, the following characterization follows by the results in [12] and [6]:

Theorem 2.1. *Suppose $d \geq 5$. Then the following are equivalent:*

(a) *C is extremal;*
(b) *C contains a planar subcurve of degree $d-1$;*
(c) *C is contained in two independent quadrics.*

The extremal curves form an interesting family of large dimension.

Theorem 2.2. *Assume $d \geq 6$ and $g \leq \frac{1}{2}(d-3)(d-4)+1$. Then the extremal curves of degree d and genus g form an irreducible generically smooth family $\mathcal{F}_{EX}$ of dimension*

$$2a + 4 + \frac{1}{2}(d-1)(d+2)$$

where $a := \binom{d-2}{2} - g$ is the maximum value of the Rao function.

For the proof see [12], where also the other values of d and g are considered.

In a subsequent paper ([15]), Nollet proved that, for $d \geq 5$, if C is not extremal, then $\rho_C(j) \leq \rho^{SE}(j)$, where $\rho^{SE} : \mathbb{Z} \to \mathbb{Z}$ is the function defined by:

$$\rho^{SE}(j) := \begin{cases} 0 & \text{if } \; j < g - \binom{d-3}{2} \\ \binom{d-3}{2} - g + j & \text{if } \; g - \binom{d-3}{2} + 1 \leq j \leq 0 \\ \binom{d-3}{2} - g + 1 & \text{if } \; 1 \leq j \leq d-3 \\ \binom{d-2}{2} - g + 1 - j & \text{if } \; d-3 \leq j \leq \binom{d-2}{2} - g \\ 0 & \text{if } \; \binom{d-2}{2} - g + 1 \leq j. \end{cases}$$

A non-degenerate curve $C \subseteq \mathbb{P}^3$ such that $\rho_C(j) = \rho^{SE}(j)$ for every $j \in \mathbb{Z}$, is called *subextremal* (see [15]). Subextremal curves are classified in [15]. We will see that the subextremal curves of degree d and genus g form a smooth irreducible family $\mathcal{F}_{SE}$ of dimension $2r + 6 + \frac{(d-2)(d+1)}{2}$, provided $r \geq 3$, where $r := \binom{d-3}{2} + 1 - g$ is the maximum value of the Rao function (cf. Theorem 5.5).

2.3. Residual sequences

Assume that C contains a planar subcurve D of degree $d - \delta \leq d$ spanning a plane H and let $\ell \in R$ be a linear form defining H. Let C' be the *residue of C with respect to H*, namely $\mathcal{I}_{C'} := \mathcal{I}_C : \mathcal{I}_H$, and let $Z \subseteq H$ be the *residue of $C \cap H$ with respect to D*, namely $\mathcal{I}_{Z,H} := \mathcal{I}_{C\cap H,H} : \mathcal{I}_{D,H}$. Let g' denote the arithmetic genus of C'.

Proposition 2.3. *With the above notation we have:*

(i) $\mathcal{I}_{Z,H}(\delta - d)$ *is isomorphic to* $\mathcal{I}_{C\cap H,H}$*, via the multiplication by an equation of* D*;*

(ii) *there exists an exact sequence (called residual sequence with respect to* H*):*

$$0 \to \mathcal{I}_{C'}(-1) \to \mathcal{I}_C \to \mathcal{I}_{Z,H}(\delta - d) \to 0,$$

where the first map is the multiplication by ℓ*;*

(iii) C' *is a curve of degree* δ*;*

(iv) Z *is either empty or zero-dimensional;*

(v) $\deg(Z) = \binom{d-\delta-1}{2} - g + g' + \delta - 1$*;*

(vi) Z *is a subscheme of* $C' \cap H$*.*

Proof. (i) - (iv) are straightforward and (vi) follows from [4], Lemma 2.8.

(v) If Z is non-empty, the residual sequence provides by considering Euler characteristics:

$$\begin{aligned} \deg Z &= \chi(\mathcal{O}_{\mathbb{P}^2}(\delta - d) - \chi(\mathcal{I}_Z(\delta - d)) \\ &= \binom{d-\delta-1}{2} + \chi(\mathcal{I}_{C'}(-1)) - \chi(\mathcal{I}_C) \\ &= \binom{d-\delta-1}{2} + \chi(\mathcal{O}_{\mathbb{P}^3}(-1)) - \chi(\mathcal{O}_{C'}(-1)) - \chi(\mathcal{O}_{\mathbb{P}^3}) + \chi(\mathcal{O}_C) \end{aligned}$$

and the conclusion follows by a straightforward computation. If Z is empty we have $\mathcal{I}_{Z,H} = \mathcal{O}_H$ and the conclusion follows by a similar argument. □

2.4. Zero-dimensional subschemes of a conic

Let $E \subseteq \mathbb{P}^2 = \mathrm{Proj}\,(S)$ be a conic where $S = K[y, z, t]$. Let $W \subseteq E$ be a zero-dimensional closed subscheme of degree r. Then it is easy to see that W satisfies:

- There is an integer b, with $0 \le b \le \frac{r-1}{2}$, such that $\partial h_W = h_b$, where $h_b : \mathbb{Z} \to \mathbb{Z}$ is the function:

$$h_b(j) := \begin{cases} 0 & \text{if} \quad j < 0 \\ 1 & \text{if} \quad j = 0 \\ 2 & \text{if} \quad 1 \le j \le b \\ 1 & \text{if} \quad b+1 \le j \le r-1-b \\ 0 & \text{if} \quad j > r-1-b. \end{cases}$$

- If $b < \frac{r-1}{2}$, then there exists a closed subscheme $W' \subseteq W$, which is collinear, of degree $r - b$.
- If $b = \frac{r}{2} - 1$, then either W is a complete intersection or again there exists a closed subscheme $W' \subseteq W$, which is collinear, of degree $r - b$.
- W is collinear if and only if $b = 0$.
- I_W can have at most three minimal generators and there are the following possibilities:
  Case 1. W is collinear. Then I_W is a complete intersection of type $(1, r)$ and its minimal free resolution has the form:

$$0 \to S(-1-r) \to S(-1) \oplus S(-r) \to I_W \to 0.$$

  Case 2. W is a complete intersection of type $(2, \frac{r}{2})$. Then its minimal free resolution has the form:

$$0 \to S(-2-\frac{r}{2}) \to S(-2) \oplus S(-\frac{r}{2}) \to I_W \to 0.$$

  Case 3. W is not a complete intersection. Then I_W has exactly three minimal generators of degree $2, b+1, a+1$, with $2 \le b+1 \le a+1$ and $a = r-b-1$, where b the integer which defines the Hilbert function of W (see above). Moreover, the minimal free resolution of I_W has the form:

$$0 \to \begin{array}{c} S(-b-2) \\ \oplus \\ S(-a-2) \end{array} \xrightarrow{\varphi} \begin{array}{c} S(-2) \\ \oplus \\ S(-b-1) \\ \oplus \\ S(-a-1) \end{array} \to I_W \to 0.$$

  The degree matrix of the Hilbert-Burch matrix representing φ is

$$\begin{bmatrix} b & a \\ 1 & a-b+1 \\ b-a+1 & 1 \end{bmatrix}.$$

  Furthermore, the ideal of W is $I_W = I(\varphi)$, where $I(\varphi)$ denotes the ideal generated by the maximal minors of the matrix.

3. Structure theorem for curves of subextremal type

In this section we consider a class of curves with large cohomology. It turns out that they have a rather particular structure. We use it to determine all occurring Rao functions among these curves.

Definition 3.1. A non-degenerate curve $C \subseteq \mathbb{P}^3$ of degree d and genus g is said to be *of subextremal type* if $d \geq 5$ and $\rho_C(j) = \binom{d-3}{2} - g + 1$ for $1 \leq j \leq d-3$.

Note that, by the results of §2, a subextremal curve is of subextremal type and that a curve of subextremal type is not extremal.

From now on, let $r := \binom{d-3}{2} - g + 1$.

For curves of subextremal type, there is the following structure theorem:

Theorem 3.2. *Let $C \subseteq \mathbb{P}^3$ be a non-degenerate curve of degree $d \geq 7$. Then the following are equivalent:*

(i) *C is of subextremal type;*
(ii) *$h^0(\mathcal{I}_C(2)) = 1$ and $h^0(\mathcal{I}_C(3)) = 5$ (that is, I_C has one minimal generator in degree 2 and one in degree 3);*
(iii) *C is contained in a unique quadric and $\partial h_\Gamma : 1\ 2\ 2\ 1\ \ldots\ 1\ 0\ \rightarrow$;*
(iv) *C contains a planar subcurve of degree $d-2$ and the residual curve C' is a planar curve of degree 2.*

Proof. (i) $\Rightarrow$ (ii). Since C is not extremal we have $\partial h_\Gamma(2) \geq 2$. It follows $h^1(\mathcal{I}_\Gamma(2)) \leq d-5$, whence $h^1(\mathcal{I}_\Gamma(j)) \leq d-3-j$ for $2 \leq j \leq d-5$ and $h^1(\mathcal{I}_\Gamma(j)) = 0$ for $j > d-5$. Hence, with an argument as in [2] (proof of Theorem 2.1, step 2), we get, for $2 \leq j \leq d-5$,

$$h^2(\mathcal{I}_C(j)) \leq \sum_{t \geq j+1} h^1(\mathcal{I}_\Gamma(t)) \leq \binom{d-3-j}{2}.$$

In particular $h^2(\mathcal{I}_C(j)) = 0$ for $j \geq d-5$.

Moreover by Riemann-Roch we have, for $1 \leq j \leq d-3$,

$$\begin{aligned} h^0(\mathcal{I}_C(j)) &= h^0(\mathcal{O}_{\mathbb{P}^3}(j)) - h^0(\mathcal{O}_C(j)) + h^1(\mathcal{I}_C(j)) \\ &= \tbinom{j+3}{3} - [dj - g + 1 + h^2(\mathcal{I}_C(j))] + r \\ &= \tbinom{j+3}{3} - dj - h^2(\mathcal{I}_C(j)) + \tbinom{d-3}{2}. \end{aligned}$$

It follows that $h^0(\mathcal{I}_C(2)) \geq 1$ and $h^0(\mathcal{I}_C(3)) \geq 5$. Since C is not extremal, we obtain $h^0(\mathcal{I}_C(2)) = 1$. Moreover, if $h^0(\mathcal{I}_C(3)) > 5$, then the exact sequence:

$$0 \rightarrow H^0(\mathcal{I}_C(2)) \rightarrow H^0(\mathcal{I}_C(3)) \rightarrow H^0(\mathcal{I}_\Gamma(3)) \rightarrow \cdots$$

provides $h^0(\mathcal{I}_\Gamma(3)) > 4$, that is $h_\Gamma(3) \leq 5$. Since $h_\Gamma(2) = 5$, this implies $5 = \deg\Gamma = d$, a contradiction.

(ii) $\Rightarrow$ (iii). From the exact sequence

$$0 \rightarrow H^0(\mathcal{I}_C(1)) \rightarrow H^0(\mathcal{I}_C(2)) \rightarrow H^0(\mathcal{I}_\Gamma(2)) \rightarrow \cdots$$

we have $h^0(\mathcal{I}_\Gamma(2)) \geq 1$ and from the exact sequence:

$$0 \rightarrow H^0(\mathcal{I}_C(2)) \rightarrow H^0(\mathcal{I}_C(3)) \rightarrow H^0(\mathcal{I}_\Gamma(3)) \rightarrow \cdots$$

we have $h^0(\mathcal{I}_\Gamma(3)) \geq 4$.

If $h^0(\mathcal{I}_\Gamma(2)) \geq 2$, then C is extremal, a contradiction. Then we must have $h^0(\mathcal{I}_\Gamma(2)) = 1$ whence $\partial h_\Gamma(1) = \partial h_\Gamma(2) = 2$.

Moreover we have $\partial h_\Gamma(3) = 1$ (as in the proof of (i) $\Rightarrow$ (ii)) and the conclusion follows.

(iii) $\Rightarrow$ (iv). Since $d \geq 7$, by [3], Corollary 4.4, C contains a subcurve of degree $d-2$ spanning a plane H. The residual sequence with respect to H is:

$$0 \to \mathcal{I}_{C'}(-1) \to \mathcal{I}_C \to \mathcal{I}_{C\cap H,H} \to 0,$$

where C' is a curve of degree 2 and the first map is the multiplication by a linear form defining H (see Proposition 2.3). The sequence above provides the exact sequence:

$$0 \to H^0(\mathcal{I}_{C'}(1)) \to H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{I}_{C\cap H,H}(2)).$$

Since $d-2 > 2$, we get $h^0(\mathcal{I}_{C\cap H,H}(2)) = 0$, whence $h^0(\mathcal{I}_{C'}(1)) = h^0(\mathcal{I}_C(2)) = 1$; thus C' is a planar curve.

(iv) $\Rightarrow$ (i). Let D be the planar subcurve of degree $d-2$. Let H be the plane that is spanned by D. Let $Z \subseteq H$ be the residual scheme of $C \cap H$ with respect to D. Then by Proposition 2.3(iv) (with $\delta = 2$ and $g' = 0$) we have $\deg Z = r$ and the residual sequence with respect to H becomes:

$$0 \to \mathcal{I}_{C'}(-1) \to \mathcal{I}_C \to \mathcal{I}_{Z,H}(2-d) \to 0.$$

Since C' is a planar curve of degree 2 we have $h^1(\mathcal{I}_{C'}(t)) = 0$ for all $t \in \mathbb{Z}$ and $h^2(\mathcal{I}_{C'}(t)) = 0$ for $t \geq 0$. Then, for $j \geq 1$ we get:

$$h^1(\mathcal{I}_C(j)) = h^1(\mathcal{I}_{Z,H}(2-d+j)).$$

If $Z \neq \emptyset$, then $h^1(\mathcal{I}_{Z,H}(t)) = \deg(Z) = r$ for $t \leq -1$, whence $h^1(\mathcal{I}_C(j)) = r$ for $1 \leq j \leq d-3$.

If $Z = \emptyset$, then $h^1(\mathcal{I}_{Z,H}(t)) = h^1(\mathcal{O}_H(t)) = 0$ for every t, thus C is arithmetically Cohen-Macaulay. □

Remark 3.3. Theorem 3.2 is false without the assumption $d \geq 7$. Indeed consider the following examples:

(i) Let C be a curve of type (1,4) on a smooth quadric Q. Then $d = 5$ and $g = 0$, whence $r = 2$. A straightforward calculation shows that $\rho_C(1) = \rho_C(2) = 2$, which implies that C is of subextremal type. On the other hand it is easy to see that (ii) of Theorem 3.2 does not hold for this curve.

(ii) Let C be a curve of type (1,5) on a smooth quadric Q. Then it is easy to see that $\partial h_\Gamma : 1\ 2\ 2\ 1\ 0 \to$, whence (iii) of Theorem 3.2 holds. But it can be shown by direct calculations that $\rho_C(1) = 3$ and $\rho_C(2) = 4$, whence C is not of subextremal type.

The next Corollary summarizes properties of curves of subextremal type which follow from Theorem 3.2 and its proof. We state them here for later use.

Corollary 3.4. *Let C be a curve of subextremal type of degree $d \geq 7$. Then we have:*

(i) *C contains a planar subcurve D of degree $d-2$ spanning a plane H;*

(ii) *the residual exact sequence with respect to H is*

$$0 \to \mathcal{I}_{C'}(-1) \to \mathcal{I}_C \to \mathcal{I}_{Z,H}(2-d) \to 0 \tag{3.1}$$

where C' a curve of degree 2 spanning a plane H' and $Z \subseteq H$ is a closed 0-dimensional subscheme with $\deg Z = r$;

(iii) *$h^1(\mathcal{I}_C(j)) = h^1(\mathcal{I}_{Z,H}(d-2+j))$ for $j \geq 0$;*

(iv) *C is contained in a unique quadric Q which is either the union of H and H', if $H \neq H'$, or it is the double plane $2H$, if $H = H'$.*

(v) $Z \subseteq C' \cap H$. *Hence if* $Z \neq \emptyset$ *we have either* $h^0(\mathcal{I}_{Z,H}(1)) \neq 0$ *or* $h^0(\mathcal{I}_{Z,H}(1)) = 0$ *and* $h^0(\mathcal{I}_{Z,H}(2)) \neq 0$.

(vi) $Z = \emptyset$ *if and only if* C *is arithmetically Cohen-Macaulay.*

Now we completely describe the Rao functions of curves of subextremal type.

Theorem 3.5. *Let* C *be a curve of subextremal type of degree* $d \geq 7$ *that is not arithmetically Cohen-Macaulay. Then we have:*

(i) *the Rao function of* C *is symmetric, namely:*

$$\rho_C(j) = \rho_C(d-2-j) \quad \text{for all } j \in \mathbb{Z};$$

(ii) *there is an integer* b, $0 \leq b \leq \frac{r-1}{2}$, *such that* $\rho_C = \rho_b$, *where* $\rho_b : \mathbb{Z} \to \mathbb{Z}$ *is the function defined by:*

$$\rho_b(j) = \begin{cases} \rho_b(d-2-j) & \text{if } \ j \leq 0 \\ r & \text{if } \ 1 \leq j \leq d-3 \\ r-1 & \text{if } \ j = d-2 \\ r-1-2(j-d+2) & \text{if } \ d-2 \leq j \leq d-2+b \\ r-1-b-j+d-2 & \text{if } \ d-2+b \leq j \leq d+r-3-b \\ 0 & \text{if } \ j \geq d+r-2-b; \end{cases}$$

(iii) *let* Z *be the* 0*-dimensional subscheme defined in Corollary 3.4. Then the following conditions are equivalent:*

(a) C *is subextremal;*

(b) Z *is collinear;*

(c) $b = 0$*;*

(d) $\rho_C(d+r-3) > 0$*;*

(iv) *if the unique quadric containing* C *is reduced, then* C *is subextremal.*

(v) C *is minimal in its biliaison class if and only if* C *is not subextremal.*

Proof. (i) Let Q be the unique quadric containing C (see Corollary 3.4). If Q is a double plane the symmetry follows from [9], Corollary 6.2.

If Q is reduced, then by Corollary 3.4 we have $Q = H \cup H'$, where H is the plane containing the planar subcurve of degree $d-2$ of C and $H' \neq H$ is the plane of C'. Then Z is contained in the line $H \cap H'$ by Corollary 3.4 and hence the homogeneous ideal $I_{Z,H}$ has a minimal generator of degree 1. From the residual exact sequence with respect to H it is not difficult to see that C is contained in a surface F of degree $d-1$ with no common components with Q. Let E be the curve linked to C by the complete intersection $Q \cap F$. By liaison (see e.g. [13])) one has: $d' := \deg E = d-2 \geq 5$, $g' := p_a(E) = g-(d-3)$, $\rho_E(j)) = \rho_C(d-3-j))$ for all $j \in \mathbb{Z}$. It follows that $\rho_E(0) = r > \frac{1}{2}(d'-3)(d'-4)+1-g'$, whence E is extremal by Nollet's bound ([15]). The conclusion follows by the definition of extremal curves.

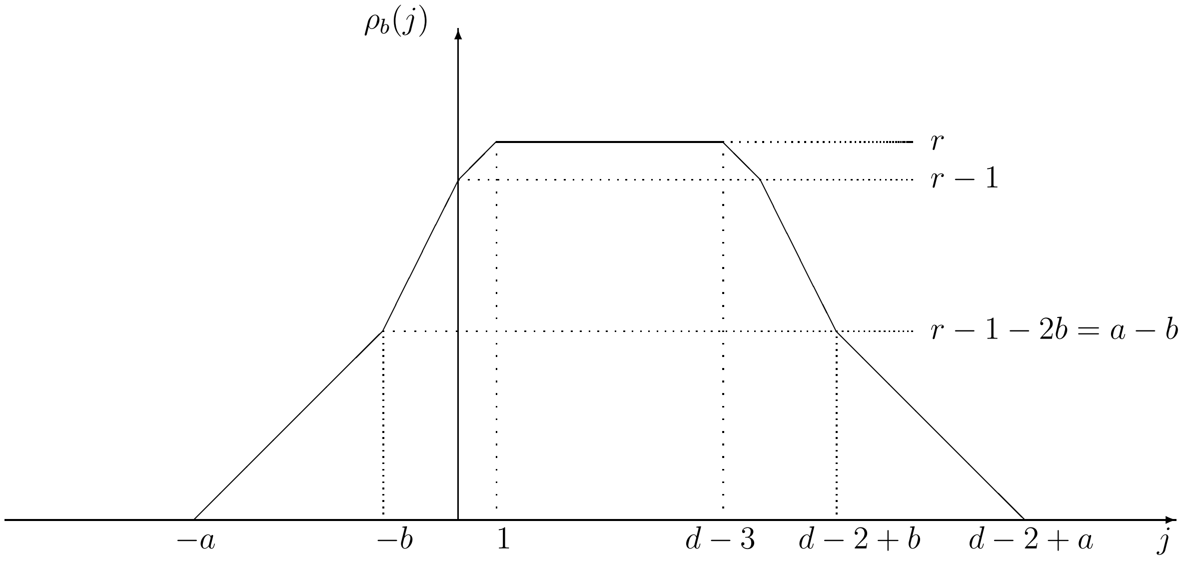
(ii) By (i) we may assume $j \geq 0$. For such values of j we have, by Corollary 3.4(iii), $h^1(\mathcal{I}_C(j)) = h^1(\mathcal{I}_Z(d-2+j)) = r - h_Z(2-d-j)$, whence $\rho_C(j) = \rho^{SE}(j)$ for $j \geq 0$ and C is subextremal by (i). The conclusion follows easily by Corollary 3.4(v) and §2.

(iii) It can be proved by an argument similar to the previous one.

(iv) If C is contained in a reduced quadric, then Z is collinear (see proof of (i)) and the conclusion follows from (iii).

(v) If C is subextremal, then it is not minimal. Conversely assume C is not minimal; then C lies in a double plane by (iv) and Z is not collinear by (iii). Hence C' is a curve of minimal degree containing Z, whence C is minimal by [9], Corollary 7.3. □

Remark 3.6. (i) The function ρ_b of Theorem 3.5, for $j \geq d-2$, decreases by 2 for b steps and then by 1 until it vanishes. See the following picture, where we put $a := r - b - 1$.



Note that a and b are the numbers introduced in §2.4 to describe the Hilbert function and the minimal free resolution of $I_{Z,H}$.

(ii) We will see that for every triple (d, g, b) of integers such that $d \geq 7,\ g \leq \binom{d-3}{2} + 1$, and $0 \leq b \leq \frac{r-1}{2}$, there exists a curve C of subextremal type having Rao function ρ_b, as prescribed by Theorem 3.5(ii). The equations of one such curve are described in Theorem 4.1 (cf. Remark 4.2); the resulting families are studied in Theorem 5.4.

4. The ideal and numerical characters of a curve of subextremal type

In this section we describe information about curves of subextremal type that we need for studying their families. At first, we focus on curves that are not subextremal.

Let C be a curve of subextremal type that is neither subextremal nor arithmetically Cohen-Macaulay. Assume that $d \geq 7$ and $\rho_C = \rho_b$ (cf. Theorem 3.5) and set $a := r - b - 1$. Using the notation of Corollary 3.4, recall that C is contained in a double plane $2H$ and $C' \subseteq D$. We may assume $H := \{x = 0\}$. We identify H with $\mathbb{P}^2$ with coordinates y, z, t and we set $I_{C'} = (\phi, x)$, where $\phi \in S := R/xR$ is a form of degree 2 and $I_D = (\phi h, x)$, for a suitable form $h \in S$ of degree $d - 4$.

The following result provides a minimal set of generators of the homogeneous ideal of C and a minimal presentation of M_C. Recall that a *Koszul module* is a graded R-module $R/(f_1, f_2, f_3, f_4)(t)$ where $t \in \mathbb{Z}$ and f_1, f_2, f_3, f_4 is a regular sequence.

Theorem 4.1. *With the above notation and assumptions we have:*

(i) *If Z is a complete intersection (namely $I_{Z,H} = (\psi, \phi)$), then*

$$I_C = (x^2, x\phi, \phi^2 h, \psi\phi h + xF)$$

where $F \in S$ is a form of degree $d-3+\frac{r}{2}$ such that the ideal $(\psi, \phi, F)S$ is irrelevant.

Moreover

$$M_C \cong [R/(x, \psi, \phi, F)](d-2).$$

In particular M_C is a Koszul module.

(ii) *If Z is not a complete intersection, possibly after a suitable choice of coordinates and bases, we may assume*

$$A := \begin{bmatrix} p & y & m \\ q & n & l \end{bmatrix}$$

to be the transpose of a Hilbert-Burch matrix of $I_{Z,H}$ where $\phi = \begin{vmatrix} y & m \\ n & l \end{vmatrix}$, $\deg p = b$, and $\deg q = a$. Then

$$I_C = \left(x^2, x\phi, \phi^2 h, \phi h \begin{vmatrix} p & m \\ q & l \end{vmatrix} + x \begin{vmatrix} m & F \\ l & G \end{vmatrix}, \phi h \begin{vmatrix} p & y \\ q & n \end{vmatrix} + x \begin{vmatrix} y & F \\ n & G \end{vmatrix}\right)$$

where $F, G \in S$ are forms such that the 2×2 minors of the homogeneous matrix

$$M := \begin{bmatrix} p & y & m & F \\ q & n & l & G \end{bmatrix}$$

generate an irrelevant ideal in S, $\deg F = b + d - 3, \deg G = a + d - 3$, and $h \in K[y, z, t]$ is a non-trivial form of degree $d - 4$.

Moreover M_C is isomorphic to the cokernel of the map

$$\begin{matrix} R(b-2) \\ \oplus \\ R(a-2) \end{matrix} \oplus \begin{matrix} R(-1) \\ \oplus \\ R(b-2) \\ \oplus \\ R(a-2) \end{matrix} \oplus R(-d+2) \to \begin{matrix} R(b-1) \\ \oplus \\ R(a-1) \end{matrix}$$

defined by the matrix $[xE_2, M]$, where E_2 is the 2×2 identity matrix.

In particular M_C is minimally generated by two homogeneous elements of degrees $1-a$ and $1-b$.

Proof. In [5] it is proved that a minimal system of generators of the homogeneous ideal of a curve C lying in a double plane can be expressed by using the maximal minors of the Hilbert-Burch matrix A of Z and of a certain homogeneous matrix B obtained from A by adding a suitable row and a suitable column. In particular by using the degree relations in Corollary 3.6 and Remark 4.7 of [5], we may assume in case (i) that

$$B = \begin{bmatrix} \psi & \phi & -F \\ 1 & 0 & 0 \end{bmatrix}$$

where F is a form of the required degree and in case (ii) that

$$B = \begin{bmatrix} p & y & m & F \\ q & n & l & G \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

with forms F and G of the required degrees.

The conclusion about the generators of I_C follows immediately by the above mentioned result. The expressions for M_C follow from [5], Theorem 4.1(i). □

Remark 4.2. (i) Note that conversely, each ideal that is defined as in the above theorem, is saturated and defines a curve of subextremal type. This follows from [5].

(ii) It is easy to produce equations for a specific curve with Rao function ρ_b. Given d, g one computes $r = \binom{d-3}{2} - g + 1$. Let b be an integer such that $0 \leq b \leq \frac{r}{2} - 1$. Then choose two lines $L_1 \neq L_2$ in the plane $\{x = 0\}$ and, for $i = 1, 2$, subsets $Z_i \subset L_i$ of b and $a = r - b - 1$ points, respectively, such that the union Z of Z_1, Z_2, and $L_1 \cap L_2$ consists of r points. Let A be the transpose of the Hilbert-Burch matrix of Z. Use the matrix $M = \begin{bmatrix} A & \begin{matrix} F \\ G \end{matrix} \end{bmatrix}$ where F, G are sufficiently general forms of degree $b+d-3$ and $a+d-3$, respectively, and, e.g., $h := y^{d-4}$ to create an ideal I as in Theorem 4.1. Then I defines a curve C of subextremal type with Rao function ρ_b. Indeed the Hilbert function of Z is h_b (cf. §2.4) and this implies that $\rho_C = \rho_b$ by Remark 3.6.

Corollary 4.3. *Let C be a curve of subextremal type with $d \geq 7$. Then*

(i) *ρ_C determines the structure of M_C, except when r is even and $b = \frac{r}{2} - 1$.*

(ii) *M_C is a Koszul module if and only if Z is a complete intersection.*

Proof. If C is subextremal, this follows by [15]. Otherwise, apply Theorem 4.1. □

The following theorem provides the minimal free resolution of I_C. It is a particular case of [5], Theorem 3.7. We write here only the modules (hence the Betti numbers), referring to the above mentioned result for an explicit description of the maps, which can be expressed in terms of the matrix B given in the proof of Theorem 4.1.

Proposition 4.4. *Let C be as above and set $a := r - b - 1$. Then the minimal free resolution of I_C is:*

<u>Case 1</u>: Z is a complete intersection of type $(2, \frac{r}{2})$ (thus $b = \frac{r}{2} - 1$).

$$0 \to R(-d-\tfrac{r}{2}-1) \to \begin{matrix} R(-4) \oplus R(-d-1) \\ \oplus \\ R(-\frac{r}{2}-d+1) \\ \oplus \\ R(-\frac{r}{2}-d) \end{matrix} \to \begin{matrix} R(-2) \oplus R(-3) \\ \oplus \\ R(-d) \oplus R(-\frac{r}{2}-d+2) \end{matrix} \to I_C \to 0$$

<u>Case 2</u>: Z is not a complete intersection.

$$0 \to \begin{matrix} R(-d-b-1) \\ \oplus \\ R(-d-a-1) \end{matrix} \to \begin{matrix} R(-4) \oplus R(-d-1) \\ \oplus \\ R^2(-d-b) \\ \oplus \\ R^2(-d-a) \end{matrix} \to \begin{matrix} R(-2) \oplus R(-3) \\ \oplus \\ R(-d) \oplus R(-d-b+1) \\ \oplus \\ R(-d-a+1) \end{matrix} \to I_C \to 0$$

In the above result, we left out the case of subextremal curves. For these, we have:

Proposition 4.5. *The minimal free resolution of a subextremal curve C of degree $d \geq 7$ is of the form:*

$$0 \to R(-r-d) \to \begin{matrix} R(-4) \oplus R(-d) \\ \oplus \\ R^2(-r-d+1) \end{matrix} \to \begin{matrix} R(-2) \oplus R(-3) \\ \oplus \\ R(-d+1) \oplus R(-r-d+2) \end{matrix} \to I_C \to 0.$$

Proof. This follows by applying the Horseshoe Lemma to the residual sequence in Corollary 3.4(ii). Note that the resulting free resolution is minimal because our assumption $d \geq 7$ guarantees that no cancellation is possible. $\square$

We now turn to the computation of numerical characters. Recall the following facts (see [10], Definition 2.3 and Proposition 2.6).

Definition 4.6. The *postulation character* of a curve $C \subseteq \mathbb{P}^3$ is the function γ_C defined by

$$\gamma_C := -\partial^3 h_C$$

Observe that γ_C and the Hilbert function h_C determine each other.

Corollary 4.7. *Let C be a curve of subextremal type of degree $d \geq 7$. Then we have:*

(i) $$h^2(\mathcal{I}_C(j)) = \begin{cases} \rho_C(j) - dj + g - 1 & \text{if } \ j < 0 \\ r + g - 1 & \text{if } \ j = 0 \\ \binom{d-3-j}{2} & \text{if } \ 1 \leq j \leq d-5 \\ 0 & \text{if } \ j > d-5; \end{cases}$$

(ii) $$h_C(j) = \begin{cases} dj - g + 1 + \binom{d-3-j}{2} - \rho_C(j) & \text{if } \ 0 \leq j \leq d-5 \\ dj - g + 1 - \rho_C(j) & \text{if } \ j > d-5; \end{cases}$$

(iii) *the index of speciality of C is $e = d - 5$;*

(iv) *the postulation character γ_C is given by:*

$$\gamma_C(j) = \begin{cases} 0 & \text{if } \ j < 0 \\ -1 & \text{if } \ 0 \leq j \leq 1 \\ 0 & \text{if } \ j = 2 \\ 1 & \text{if } \ j = 3 \\ 0 & \text{if } \ 4 \leq j \leq d-1 \\ \partial^3 \rho_C & \text{if } \ j \geq d; \end{cases}$$

in particular, it depends only on ρ_C.

Proof. Adopt the notation in Corollary 3.4.

(i) From the residual exact sequence with respect to H we have the exact sequence:

$$\cdots \to H^2(\mathcal{I}_{C'}(j-1)) \to H^2(\mathcal{I}_C(j)) \to H^2(\mathcal{I}_{Z,H}(2-d+j)) \to H^3(\mathcal{I}_{C'}(j-1)) \to \ldots.$$

First assume $j \geq 1$. Then $H^2(\mathcal{I}_{C'}(j-1)) = 0$ and $H^3(\mathcal{I}_{C'}(j-1)) = H^3(\mathcal{O}_{\mathbb{P}^3}(j-1)) = 0$, whence $h^2(\mathcal{I}_C(j)) = h^2(\mathcal{I}_{Z,H}(2-d+j)) = h^2(\mathcal{O}_{\mathbb{P}^2}(2-d+j)) = h^0(\mathcal{O}_{\mathbb{P}^2}(d-5-j))$ and the conclusion follows in this case.

Assume now $j \leq 0$. We have $h^0(\mathcal{O}_C(j)) = dj - g + 1 + h^2(\mathcal{I}_C(j))$ and since

$$h^0(\mathcal{O}_C(j)) = \begin{cases} \rho_C(j) & \text{if } \; j < 0 \\ \\ \rho_C(0) + 1 & \text{if } \; j = 0 \end{cases}$$

the conclusion follows, recalling that $\rho_C(0) + 1 = r$ by Theorem 3.5 (ii).

(ii) We have:

$$h_C(j) = h^0(\mathcal{O}_C(j)) - \rho_C(j) = dj - g + 1 + h^2(\mathcal{I}_C(j)) - \rho_C(j).$$

If $j \geq 1$ the conclusion follows from (i). On the other hand we have $h_C(0) = 1$ and the conclusion follows from the definition of r, since $\rho_C(0) = r - 1$.

(iii) is an immediate consequences of (i) (ii) and the definitions.

(iv) Follows from (ii) and a straightforward calculation. □

Corollary 4.8. *Let $C \subseteq \mathbb{P}^3$ be a curve and assume that γ_C coincides with the postulation character of a curve of subextremal type having the same degree and genus as C.*

Then C is of subextremal type and ρ_C depends only on γ_C.

Proof. By definition γ_C determines h_C. The conclusion follows from Theorem 3.2 (ii) and Corollary 4.7(iv). □

Remark 4.9. Recall that the *spectrum* of a curve C is the function $\ell_C(j) := \partial^2 h^0(\mathcal{O}_C(j)) = \partial^2 h^2(\mathcal{I}_C(j))$ (see [16]) and that the *speciality character* of C is the function $\sigma_C := \partial \ell_C$ (see [10], Definition 2.3). Then by Corollary 4.7 we see that if C is a curve of subextremal type then ρ_C determines ℓ_C and σ_C and conversely.

5. The family of curves of subextremal type

In this section we want to show that the curves of subextremal type of given degree and genus form a family and to describe this family.

We consider only projective families parameterized by the closed points of algebraic k-schemes. If X is a scheme and $x \in X$ we denote by $\kappa(x)$ the residue field of the local ring $\mathcal{O}_{X,x}$.

Throughout this section we fix the integers d and g and we put $r := \binom{d-3}{2} + 1 - g$. Moreover we denote by $H_{d,g}$ be the Hilbert scheme of locally Cohen-Macaulay curves of degree d and genus g. We refer to [17] and [10] for basic information about families and Hilbert schemes.

We begin with a result which is of course expected but still needs a proof.

Lemma 5.1. *Let $E \times \mathbb{P}^3 \supseteq X \to E$ be a family of curves of subextremal type of degree $d \geq 7$ and genus g and let $G \times \mathbb{P}^3 \supseteq Y \to G$ be the family of quadrics of $\mathbb{P}^3$. Then we have:*

- (a) *There is a morphism $f : E \to G$ such that $G_{f(e)}$ is the unique quadric containing X_e, for each $e \in E$ (see Theorem 3.2);*
- (b) *The subset $E^{(2)} \subseteq E$ corresponding to the curves contained in some double plane is closed.*

Proof. (a) Let $U = \operatorname{Spec}(A) \subseteq E$ be an open affine subset and set $B := A[x, y, z.t]$, the graded polynomial ring in 4 variables. Then $X_U := X \cap (\mathbb{P}^3 \times U)$ is a closed subscheme of $\mathbb{P}^3 \times U = \operatorname{Proj}(B)$. Let $I \subseteq B$ be the saturated homogeneous ideal of X_U. For each $e \in U$, $(B/I) \otimes \kappa(e)$ is the homogeneous coordinate ring of X_e, whence by Theorem 3.2 (iii) we have $\dim_{\kappa(e)}((B/I)_2 \otimes \kappa(e)) = 9$ for all $e \in U$. This implies that $(B/I)_2$ is a projective

A-module of rank 9, whence I_2 is a projective A-module of rank 1. Let $J := I_2 B$. It is a homogeneous saturated ideal defining a closed subscheme $Q_A \subseteq \mathbb{P}^3_A$ that is flat over A. By construction, $(Q_A) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\kappa(e))$ is the unique quadric containing X_e, for all $e \in U$. Now, by the universal property of the Hilbert scheme, there is a canonical morphism $f_U : U \to G$ and moreover, letting U vary in an open affine covering of E, the morphisms f_U glue together and produce the required morphism f.

(b) Since double planes correspond to a closed subset of G the conclusion follows immediately from (a). □

The next Lemma shows the existence of the family we are interested in and states some of its elementary properties.

Lemma 5.2. *Assume $d \geq 7$ and $g \leq \binom{d-3}{2} + 1$. Then we have:*

(a) *There exists a reduced locally closed subscheme $\mathcal{F}_{SET} \subset H_{d,g}$ which parameterizes the curves of subextremal type.*

(b) *There are closed subschemes $\mathcal{F}_{SE}$ and $\mathcal{F}^{(2)}_{SET}$ of $\mathcal{F}_{SET}$ which parameterize the subextremal curves and the curves of subextremal type lying in some double plane, respectively. Moreover $\mathcal{F}_{SET} = \mathcal{F}^{(2)}_{SET} \cup \mathcal{F}_{SE}$.*

Proof. By the definition of curve of subextremal type and by semicontinuity it follows that the subset corresponding to the curves of subextremal type is locally closed, hence it carries a natural structure of reduced subscheme of $H_{d,g}$. This is $\mathcal{F}_{SET}$.

(b) By Theorem 3.5, a curve C of subextremal type is subextremal if and only if $\rho_C(d + r - 1) > 0$. Hence by semicontinuity $\mathcal{F}_{SE}$ is a closed subset of $\mathcal{F}_{SET}$. Moreover the subset $\mathcal{F}^{(2)}_{SET}$ is closed by Lemma 5.1. The last assertion is clear. □

Now we are going to study the two subfamilies $\mathcal{F}_{SE}$ and $\mathcal{F}^{(2)}_{SET}$ in more detail. In particular we will show that, if $r \geq 3$, they are the two irreducible components of $\mathcal{F}_{SET}$ and have the same dimension.

We begin with $\mathcal{F}^{(2)}_{SET}$. For this we will need the following concept:

Definition 5.3. Let b be an integer satisfying $0 \leq b \leq \lfloor \frac{r-1}{2} \rfloor$. A curve C is said to be *of subextremal type b* if $\rho_C = \rho_b$ (see Theorem 3.5). Thus a subextremal curve is a curve of subextremal type 0.

Theorem 5.4. *Assume $d \geq 7$ and $r \geq 0$. Then:*

(a) *The subscheme $\mathcal{F}^{(2)}_{SET} \subset H_{d,g}$ is irreducible and has dimension $2r + 6 + \frac{(d-2)(d+1)}{2}$.*

(b) *If $r \geq 3$ there exists a stratification of $\mathcal{F}^{(2)}_{SET}$ given by the Rao function, namely the curves of subextremal type b form a non-empty irreducible subfamily $\mathcal{F}^{(2)}_{SET_b}$, which is:*

  (i) *open, generically smooth for even r and smooth for odd r, if $b = \lfloor \frac{r-1}{2} \rfloor$,*

  (ii) *smooth, locally closed of codimension 1, if $0 < b < \lfloor \frac{r-1}{2} \rfloor$,*

  (iii) *smooth, closed of codimension 1, if $b = 0$ (it parameterizes the subextremal curves that are contained in some double plane);*

Proof. We use the techniques of [9]. Let $2H$ be a fixed double plane and let $C \subseteq 2H$ be a curve of subextremal type. Then one can associate to C a flag of subschemes of H, namely $Z \subseteq C' \subseteq D$, where D is the planar subcurve of C of degree $d - 2$ (Theorem 3.2,(iv)) and Z and C' come from the exact residual sequence with respect to H (see Corollary 3.4,(ii)). Recall that $\deg Z = r$ and $\deg C' = 2$. Consider now the set of all curves of subextremal type contained in $2H$ of degree d and genus g. This coincides set theoretically with the scheme $H_{r,2,d-2}(2H)$ defined in [9], which is irreducible and generically smooth of dimension $2r+3+\frac{(d-2)(d+1)}{2}$ (see [9], Corollary 4.3).

Now by Lemma 5.1 there is a surjective morphism $f : \mathcal{F}^{(2)}_{SET} \to T$, where $T \cong \mathbb{P}^3$ parameterizes the double planes. Clearly the fibers of f are homeomorphic to $H_{r,2,d-2}(2H)$. Then (a) follows.

Now we prove (b). First of all observe that, due to the particular shape of the Rao functions (Theorem 3.5) and by semicontinuity, the subsets $\mathcal{F}^{(2)}_{SET_b}$ are locally closed (open for $b = \lfloor \frac{r-1}{2} \rfloor$, closed for $b=0$) and form a stratification of $\mathcal{F}^{(2)}_{SET}$.

To prove the remaining properties it is easy to see that, by using the morphism f defined above, it is sufficient to study the problem in a fixed double plane $2H$.

According to [9] (see Propositions 4.1 and 4.2 and their proofs) there is a smooth fibration $\pi : H_{r,2,d-2}(2H) \to D_{r,2}(H)$, where $D_{r,2}(H)$ is the flag scheme consisting of the pairs (Z, C') where $C' \subseteq H$ is a conic and Z is a locally complete intersection zero-dimensional scheme of degree r contained in C'.

For each integer b with $0 \le b \le \lfloor \frac{r-1}{2} \rfloor$, set $T_b(H) = \{(Z, C') \in D_{r,2}(H) \mid \partial h_Z = h_b\}$ (see §2) and let $\mathcal{F}^{(2)}_{SETb}(H)$ be the set of curves of subextremal type b contained in $2H$. Then $\mathcal{F}^{(2)}_{SETb}(H) = \pi^{-1}(T_b(H))$ (see proof of Corollary 3.4). Recall that $D_{r,2}$ is irreducible and generically smooth of dimension $r+5$ (this follows from [1]: see [9], proof of Proposition 4.2, for the idea of the proof). Moreover by [7], Theorem 2.4, the stratification of $D_{r,2}$ given by the Betti numbers has locally closed smooth irreducible strata. Now by Proposition 4.4 the Betti numbers of Z depend only on b if $0 \le b < \frac{r}{2}$, whence $\mathcal{F}^{(2)}_{SET_b}(H)$ is smooth and irreducible for such values of b. If r is even and $b = \frac{r}{2}-1$ we have two different strata, so we can only say that $\mathcal{F}^{(2)}_{SET_{\frac{r}{2}-1}}(H)$ is irreducible and generically smooth.

This proves (i) and the smoothness statements in (ii) and (iii).

Now we compute the dimensions of the strata. As above it is sufficient to study the problem for a fixed plane H.

If $b=0$, then Z is a complete intersection $(1,r)$. These complete intersections form a smooth irreducible family of dimension $2+r$, as it is easily seen, and hence $\dim T_0 = 4+r = \dim D_{r,2}(H) - 1$. It follows that $\mathcal{F}^{(2)}_{SET_0}(H)$ is smooth and $\dim \mathcal{F}^{(2)}_{SET_0}(H) = \dim H_{r,2,d-2}(2H) - 1 = \dim \mathcal{F}^{(2)}_{SET} - 1$. This completes the proof of (iii).

It remains to compute $\dim(\mathcal{F}^{(2)}_{SET_b})$ for $0 < b < \lfloor \frac{r-1}{2} \rfloor$. For this it is sufficient to compute $\dim(T_b(H))$. Now for b in the given range we have $r \ge 5$ and $r-b \ge 3$. Hence $\partial h_Z = h_b$ if and only if there is a line $L \subseteq H$ such that $\deg(L \cap Z) = r-b$ whence the unique conic containing Z is reducible.

This implies that $\dim(T_b(H)) \le \dim(D_{r,2}(H)) - 1 = r+4$. Now the reduced schemes $Z \subseteq H$ consisting of $r-b$ points on a line and b points on a different line form a non-empty open subset $U \subseteq T_b(H)$. It is easy to see that $\dim(U) = r+4$ whence $\dim(T_b(H)) = r+4$. It follows that $\dim \mathcal{F}^{(2)}_{SET_b} = \dim \mathcal{F}^{(2)}_{SET} - 1$ for b in the given range. This completes the proof of (ii). □

Now we turn our attention to the family of subextremal curves $\mathcal{F}_{SE}$.

Theorem 5.5. *If $d \ge 7$ the subscheme $\mathcal{F}_{SE} \subset H_{d,g}$ is irreducible. If moreover $r \ge 3$, then $\mathcal{F}_{SE}$ is smooth of dimension $2r+6+\frac{(d-2)(d+1)}{2}$*

Proof. Let C be a subextremal curve. By definition, its Rao function ρ_C depends only on d and g (see §2). Moreover, by Corollary 4.7 we have that γ_C can be computed in terms of ρ_C, hence it is independent of the particular curve C. It follows that the subextremal curves are parameterized by the subscheme $H_{\gamma,\rho} \subseteq H_{d,g}$, as defined in [10], Definition VI.3.14, where $\rho := \rho_C$ and $\gamma := \gamma_C$. Then we have $\mathcal{F}_{SE} = (H_{\gamma,\rho})_{red}$.

Now we show that $H_{\gamma,\rho}$ is irreducible. To this end we use some ideas from liaison theory. Let

$$0 \to F \to N \oplus G \to I_C \to 0$$

be the minimal N-type resolution of the subextremal curve C where F, G are free R-modules of smallest possible rank. Then N is the second syzygy module of the Hartshorne-Rao module M_C of C (see [10]). Since M_C is a Koszul module, we know the minimal free resolution of N. Using the mapping cone procedure, the above sequence provides a free resolution of I_C. Comparing with the graded Betti numbers of C (cf. Proposition 4.5), we see that we must have $G = R(-2)$ and $F = R(r-4) \oplus R(-3) \oplus R(-d+1)$. Hence, the corresponding modules in the minimal N-type resolutions of each two subextremal curves C, $\widetilde{C}$ are isomorphic. Thus we conclude as in Step (IV) of the proof of [14], Theorem 7.3, that C and $\widetilde{C}$ belong to a flat family whose members belong to $H_{\gamma,\rho}$ and that is parameterized by an open subset of $\mathbb{A}^1$. The irreducibility of $H_{\gamma,\rho}$ follows.

Now assume $r \geq 3$. By Theorem 5.4(iii), the subextremal curves contained in some double plane form an irreducible family of dimension $2r + 5 + \frac{(d-2)(d+1)}{2}$. Since there are subextremal curves that are not contained in a double plane (for example, perform a basic double linkage on a reduced reducible quadric starting from an extremal curve of degree $d-2$ and genus $g-d+3$), we have

$$\dim \mathcal{F}_{SE} \geq 2r + 6 + \frac{(d-2)(d+1)}{2}.$$

Hence, to conclude our proof it is sufficient to show that the tangent space of $H_{\gamma,\rho}$ at every closed point t has dimension $t_{\gamma.\rho} = 2r + 6 + \frac{(d-2)(d+1)}{2}$.

Let C be the curve corresponding to t and let M be the Rao module of C. Then by [10], Theorem IX.4.2, the dimension of the tangent space of $H_{\gamma,\rho}$ at t is

$$t_{\gamma,\rho} = \delta_\gamma + \epsilon_{\gamma,\rho} - \dim_k(\mathrm{Hom}(M,M)^0) + \dim_k(\mathrm{Ext}^1(M,M)^0),$$

where δ_γ and $\epsilon_{\gamma,\rho}$ are the number defined in [10], ch. IX, 3.1.

The calculations are lengthy but elementary and make use of the assumption $r \geq 3$. First of all one computes γ from Corollary 4.7, and from this one gets

$$\delta_\gamma = \frac{(d-2)(d+1)}{2} + 8 - r.$$

Next, from ρ and γ one finds $\epsilon_{\gamma,\rho} = r - 4$. Since M is a Koszul module, one has that $\dim_k(\mathrm{Hom}(M,M)^0) = 1$ and from ρ and [10], ch. IX, example 6.1, one gets $\dim_k(\mathrm{Ext}^1(M,M)^0) = 2r+3$. It follows

$$t_{\gamma,\rho} = 2r + 6 + \frac{(d-2)(d+1)}{2}.$$

This completes the proof. □

From Lemma 5.2 and Theorems 5.4 and 5.5 we have immediately:

Corollary 5.6. *Assume $d \geq 7$ and $r \geq 3$. The reduced subscheme $\mathcal{F}_{SET} \subset H_{d,g}$ is of pure dimension $2r+6+\frac{(d-2)(d+1)}{2}$ and its irreducible components are $\mathcal{F}_{SET}^{(2)}$ and $\mathcal{F}_{SE}$. Moreover $\left(\mathcal{F}_{SET}^{(2)} \cap \mathcal{F}_{SE}\right)_{red} = \mathcal{F}_{SET_0}^{(2)}$.*

Remark 5.7. In the previous results we have made the assumption $r \geq 3$. If $1 \leq r \leq 2$ then $\mathcal{F}_{SET} = \mathcal{F}_{SE}$ by Theorem 3.5 and the stratification of Theorem 5.4(b) is trivial. By Theorem 5.4(a) and Theorem 5.5 we have that $\mathcal{F}_{SET}$ is irreducible and has dimension $\dim(\mathcal{F}_{SET}) > 2r + 6 + \frac{(d-2)(d+1)}{2}$. The interested reader might carry out the calculation of $t_{\gamma,\rho}$ as in the proof of Theorem 5.5 and get some more precise information.

6. Two components of the Hilbert scheme

In this section we show that, if $d \geq 7$ and $g < 0$, then the closures of $\mathcal{F}_{SE}$ and of $\mathcal{F}^{(2)}_{SET}$ in $H_{d,g}$ are, topologically, irreducible components of $H_{d,g}$. We use the same notation as in the previous section.

We begin with a geometrical description of the "general subextremal curve."

Theorem 6.1. *Assume $d \geq 7$ and $r \geq 3$. Then:*

(a) *If C' is an extremal curve of degree $d-1$ and genus $g-1$ and L is a 2-secant line of C' then $C := L \cup C'$ is a sub-extremal curve of degree d and genus g.*

(b) *There is a non-empty open set $U \subseteq \mathcal{F}_{SE}$ such that every curve $C \in U$ is as in* (a).

(c) *There is a non-empty open set $U' \subseteq U$ such that every $C \in U'$ has a scheme-theoretical decomposition*

$$C = D \cup Y \cup L$$

where D is a smooth planar curve of degree $d-3$ spanning a plane H, $Y \not\subseteq H$ is a double line whose support lies in H and $L \not\subseteq H$ is a 2-secant line to Y. Moreover, the arithmetic genus of Y satisfies $g_Y \leq -r$ and C_{red} is a curve of degree $d-1$ of maximal genus.

Proof. (a) From §2.2 it follows that the maximum of $\rho_{C'}$ is r and C' contains a subcurve P of degree $d-2$ spanning a plane H and that the residual curve of C' with respect to H is a line ℓ. It is clear that $\deg C = d$ and by the genus formula for the union of two curves it follows that $p_a(C) = g$. Since $\deg P \geq 3$ we have $L \not\subseteq H$, whence C contains a planar subcurve of degree $d-2$, namely P, but it does not contain a planar subcurve of degree $d-1$.

Now by [12], Proposition 0.6 and our numerical assumptions, we have that $\ell \subseteq P$ and $C = D' \cup Q$, where Q is a multiple line supported by ℓ and $D' \not\supseteq \ell$. Then it is clear that L must be a secant line of Q and, in particular, L meets ℓ. This implies that the residual curve of C with respect to H is the planar degree 2 curve $\ell \cup L$. Hence C is of subextremal type by Theorem 3.2. Moreover since $L \not\subseteq H$ the unique quadric containing C is reduced whence C is subextremal by Theorem 3.5. This proves (a)

(b) Now we want to show that the subextremal curves constructed above form a family, and we want to compute its dimension.

Let $E \times \mathbb{P}^3 \supseteq X \to E$ be the family of extremal curves of degree $d-1$ and genus $g-1$ and let $G \times \mathbb{P}^3 \supseteq Y \to G$ be the Grassmannian of lines of $\mathbb{P}^3$. Recall that $\dim E = 2r + 4 + \frac{(d-2)(d+1)}{2}$ by Theorem 2.2.

Now the family

$$E \times G \times \mathbb{P}^3 \supseteq (X \times G) \cap (E \times Y) \to E \times G$$

parameterizes bijectively the intersections $X_e \cap Y_g$ and by Chevalley's theorems there is a locally closed subset $V \subseteq E \times G$ such that $(e, g) \in V$ if and only if $\mathrm{length}(X_e \cap Y_g) = 2$ (that is if and only if Y_g is a 2-secant line of X_e).

For any $e \in E$ let H_e be the plane containing the planar subcurve of X_e of degree $d-2$ and let ℓ_e be the residual line of X_e with respect to H_e. Let Q_e be the largest subcurve of X_e supported by ℓ_e. Then, as we have seen above, Y_g is a 2-secant line of X_e if and only if it is a 2-secant line of Q_e. Now it easily follows that the fibers of the projection $V \to E$ have dimension 2, whence $\dim V = \dim E + 2 = 2r + 6 + \frac{(d-2)(d+1)}{2}$.

Consider now the family

$$E \times G \times \mathbb{P}^3 \supseteq (E \times Y) \cup (X \times G) \xrightarrow{\varphi} E \times G.$$

It is easy to see that it parameterizes bijectively the schemes $X_e \cup Y_g$. Thus, it follows that the subextremal curves constructed in (a) are exactly the curves of the family $\varphi^{-1}(V) \to V$.

By the universal property of the Hilbert scheme there is an injective morphism $\Phi : V \to H_{d,g}$ and $\Phi(V) \subseteq \mathcal{F}_{SE}$ by (a). Moreover $\Phi(V)$ is constructible and since $\dim V = \dim \mathcal{F}_{SE}$ and $\mathcal{F}_{SE}$ is irreducible by Theorem 5.5 it follows that $\Phi(V)$ contains a non-empty open subset U of $\mathcal{F}_{SE}$ and the conclusion follows.

(c) By [12], Proposition 0.6 it follows that there is a non-empty open subset $E' \subseteq E$ such that every $C' \in E'$ has a scheme-theoretical decomposition $C' = D \cup Y$, where D is a planar smooth curve of degree $d-3$ spanning a plane H and Y is a double line whose support lies in H. Let $\pi : E \times G \to E$ be the projection. Then one shows as above that the image of $\pi^{-1}(E') \cap V$ in $H_{d,g}$ contains a non-empty open subset U' of $\mathcal{F}_{SE}$ with the required properties.

The genus of Y can be easily bounded by using the formula $g = p_a(D) + p_a(Y) + \operatorname{length} D \cap Y - 1$, observing that $\operatorname{length}(D \cap Y) \geq d-3$.

Finally if $C \in U'$ then $C_{red} = D \cup Y_{red} \cup L$ is a curve of degree $d-1$ and of maximal genus, being the union of a planar curve and a line meeting it in a scheme of length 1 (see [8]). □

Now we can deal with $\mathcal{F}_{SE}$.

Theorem 6.2. *Assume $d \geq 7$ and $g < 0$. Then the closure of $\mathcal{F}_{SE}$ in $H_{d,g}$ is, topologically, an irreducible component of $H_{d,g}$.*

Proof. Observe first that our numerical assumptions imply $r \geq 8$, whence, in particular, Theorem 6.1 applies.

We follow some ideas from [12], proof of Proposition 3.6. By Theorem 5.5, there is an irreducible component $\mathcal{F}$ of $H_{d,g}$ containing $\mathcal{F}_{SE}$ We want to show that $\mathcal{F} = \overline{\mathcal{F}_{SE}}$. We argue by contradiction, assuming that $\mathcal{U} := \mathcal{F} \setminus \overline{\mathcal{F}_{SE}} \neq \emptyset$.

We use the scheme-theoretical decomposition $C_0 = D \cup Y \cup L$ of a general $C_0 \in \mathcal{F}_{SE}$ given in Theorem 6.1 (c).

Fix C_0 as above and let $C \in \mathcal{U}$. Then there is a flat family $X \to T$, where T is a smooth connected curve, and points $t_0, u \in T$ such that $C_0 = X_{t_0}$, $C = X_u$, and $X_t \in \mathcal{U}$ for $t \in T \setminus \{t_0\}$.

By our assumption on g every curve $C \in \mathcal{U}$ is non-integral, hence it is either non-reduced or reduced and reducible. We consider the two cases separately.

<u>Case 1</u>. Assume that C is non-reduced. Then X is non reduced and we set $X' := X_{red}$. Since T is a smooth curve the family $X' \to T$ is flat. Moreover $X'_t = (X_t)_{red} \neq X_t$ for general $t \in T$ and $(C_0)_{red} \subseteq X'_{t_0} \subsetneq C_0$. By the particular shape of C_0 we have $\deg(C_0)_{red} = \deg C_0 - 1 = d-1$, whence $(C_0)_{red} = X'_{t_0}$. By flatness we have that X'_t is a curve of degree $d-1$ and maximal genus, as $(C_0)_{red}$ is such a curve. It follows that X'_t is the union of a planar curve P_t and of a line ℓ_t meeting P_t in a scheme of length 1 (see

[8]). Now by degree reasons X_t contains a double line Y_t. If $(Y_t)_{red} \subseteq P_t$ we have that $X_t \in \mathcal{F}_{SE}$ whence $C \in \mathcal{F}_{SE}$, a contradiction. So we have $(Y_t)_{red} = \ell_t$. We want to show that this leads again to a contradiction.

Assume first that P_t is integral. Then X'_t has two irreducible components, namely P_t and ℓ_t. It follows that X' has two irreducible components X'_1 and X'_2 corresponding to P_t and ℓ_t, respectively. Since X has no embedded components, it follows that it has exactly two irreducible components X_1 and X_2 that are, by degree reasons, topologically equal to X'_1 and X'_2, respectively. This implies that $(X_2)_{t_0}$ is a sub-curve of C_0 supported by L, hence $(X_2)_{t_0} = L$. But this is a contradiction because the families $X_i \to T$ are flat, being T a smooth curve, while $\deg(X_2)_{t_0} \neq \deg(X_2)_t$. Hence, P_t is not integral.

It follows that every general $C \in \mathcal{U}$ has a scheme-theoretical decomposition $C = P \cup W$, where P is a non-integral planar curve of degree $d-2$ and W is a double line whose support meets P but does not lie in the plane spanned by P. In particular $\epsilon := \operatorname{length} W \cap P \in \{1, 2\}$.

Now since $g = p_a(P) + p_a(W) + \epsilon - 1$ we get $-r + 1 \geq p_a(W) \geq -r$, whence, in particular, $p_a(W) \leq -2$. It also follows that the double lines W move in a family of dimension $5 - 2p_a(W) \leq 5 + 2r$ (see [12], Theorem 4.1). Since the non-integral planar curves of degree $d - 2$ move in a family of dimension $\frac{1}{2}d(d-3) + 5$ we have $\dim \mathcal{F} = \dim \mathcal{U} \leq 5 + 2r + \frac{1}{2}d(d-3) + 5 \leq 2r + 6 + \frac{1}{2}(d-2)(d+1) = \dim \mathcal{F}_{SE}$ where the last equality is due to Theorem 5.5. This is a contradiction, and the conclusion follows.

Case 2. Assume that C is reduced. Then X is reduced and reducible, namely there is a proper scheme decomposition $X = X_1 \cup X_2$ and the families $X_i \to T$ are flat since T is a smooth curve. Moreover we have, set-theoretically, $(X_1)_{t_0} \cup (X_2)_{t_0} = C_0$. Up to interchanging X_1 and X_2 we have three possibilities, namely:

(i) $((X_1)_{t_0})_{red} = L$ and $((X_2)_{t_0})_{red} = Y_{red} \cup D$
(ii) $((X_1)_{t_0})_{red} = Y_{red}$ and $((X_2)_{t_0})_{red} = L \cup D$
(iii) $((X_1)_{t_0})_{red} = D$ and $((X_2)_{t_0})_{red} = Y_{red} \cup L$

If (i) holds then $(X_1)_{t_0} = L$ whence, by degree reasons, $(X_2)_{t_0} = Y \cup D$. It follows that $(X_2)_{t_0}$ is an extremal curve of degree $d-1$ and genus $g-1 < 0$. Then $(X_2)_t$ is non-integral, hence it is extremal by [12], Proposition 3.6. Now (as in the proof of Theorem 6.1, (a)) r is the maximum of the Rao function of $(X_2)_{t_0}$ and hence of $(X_2)_t$. Since $r \geq 8$, $(X_2)_t$ is not reduced (see [12], Proposition 0.6), whence X is not reduced, a contradiction.

If (ii) holds then $(X_2)_{t_0} = L \cup D$, whence $(X_1)_{t_0} = Y$. Then $(X_1)_t$ is a curve of degree 2. Moreover by Theorem 6.1,(c) we have $p_a((X_1)_t) = p_a(Y) \leq -r \leq -3$, which implies that $(X_1)_t$ is not reduced, whence X is not reduced, again a contradiction.

If (iii) holds we get, arguing as above, $(X_2)_{t_0} = L \cup Y$. It is easy to show that $p_a(L \cup Y) \leq -r + 1 \leq -7$. Since $\deg(L \cup Y) = 3$ we get $p_a((X_2)_t) \leq -\deg((X_2)_t)$, whence X_t is not reduced. As above, it follows that X is not reduced, a contradiction. □

Now we consider $\mathcal{F}^{(2)}_{SET}$. Our strategy is similar to the previous one for $\mathcal{F}_{SE}$. We begin with a geometric description of the general curve in $\mathcal{F}^{(2)}_{SET}$.

Lemma 6.3. *Assume $d \geq 7$. Then there is a non-empty open set $U \subseteq \mathcal{F}^{(2)}_{SET}$ such that every $C \in U$ admits a scheme-theoretical decomposition $C = Y \cup E$, where E is a smooth planar curve of degree $d - 4$ contained in a plane H and Y is a curve of degree 4 whose support is a smooth conic contained in H. Moreover, $p_a(Y) \leq -r + 1$.*

Proof. Let H be a fixed plane. Then by [9] there is a morphism $\sigma : H_{r,2,d-2}(2H) \to D_{r,2,d-2}(H)$, where $D_{r,2,d-2}(H)$ is the flag scheme parameterizing the triples (Z, C', D)

(see proof of Theorem 5.4). The subset $D'_{r,2,d-2}(H) \subseteq D_{r,2,d-2}(H)$, where C' is a smooth conic, $D = C' \cup E$, and E is smooth, is open and non-empty, thus $\pi^{-1}(D'_{r,2,d-2}(H))$ is a non-empty open subset of $H_{r,2,d-2}(2H)$ whose curves have the required scheme-theoretical decomposition.

Let now $P \in \mathbb{P}^3$ be a point and denote by $(\mathcal{F}^{(2)}_{SET})_P$ the set of curves in $\mathcal{F}^{(2)}_{SET}$ lying in a double plane not containing P. By Lemma 5.1 the subsets $(\mathcal{F}^{(2)}_{SET})_P$, as P varies, form an open covering of $\mathcal{F}^{(2)}_{SET}$. For every $P \in \mathbb{P}^3$ fix a plane H_P not containing P. Then the projection from P induces a surjective morphism $g_P : (\mathcal{F}^{(2)}_{SET})_P \to H_{r,2,d-2}(2H_P)_{red}$ and the existence of U follows.

Let now $C = Y \cup E$ be a curve in U. Then the scheme $E \cap Y$ is zero-dimensional and $\operatorname{length}(E \cap Y) \geq 2(d-4)$. Since $g = p_a(Y) + p_a(E) + \operatorname{length}(E \cap Y) - 1$, the bound for $p_a(Y)$ is obtained by a straightforward calculation. □

Theorem 6.4. *Assume $d \geq 7$ and $g < 0$. Then the closure of $\mathcal{F}^{(2)}_{SET}$ in $H_{d,g}$ is, topologically, an irreducible component of $H_{d,g}$.*

Proof. We use the same setting and notation (with obvious modifications) as in the proof of Theorem 6.2. In particular $C_0 = E \cup Y$ will have the structure given by Lemma 6.3. Observe also that our numerical assumptions imply $r \geq 8$, whence $p_a(Y) \leq -7$.

Case 1. The general curve in $\mathcal{U}$ is not reduced. Then, Y being irreducible, we have $X'_{t_0} = E \cup Y_{red}$. Thus there are only two possibilities, namely:

(i) X_t contains a double conic; or

(ii) X_t contains a multiple line.

If (i) holds then $X_t \in \mathcal{F}^{(2)}_{SET}$ and we are done. If (ii) holds then X'_t is the union of a curve of degree $d-3$ and a line. Let $X' = X'_1 \cup X'_2$ be the corresponding decomposition. Then $(X'_2)_{t_0}$ is a line contained in $E \cup Y_{red}$, which is impossible.

Case 2. The general curve in $\mathcal{U}$ is reduced and reducible. then we have $X = X_1 \cup X_2$ with $(X_1)_{t_0} = Y$. But $p_a((X_1)_t) = p_a((X_1)_{t_0}) \leq -7 < -4 = -\deg((X_1)_t)$. Then X is not reduced, a contradiction. □

With respect to the Hilbert scheme, our results about curves of subextremal type can be summarized as follows:

Corollary 6.5. *If $d \geq 7$ and $g < 0$, then the Hilbert scheme $H_{d,g}$ has at least three components. Topologically, two components are the closures of $\mathcal{F}^{(2)}_{SET}$ and $\mathcal{F}_{SE}$ and another component is formed by the closure of $\mathcal{F}_{EX}$ that parameterizes the extremal curves. The first two components have dimension $\frac{3}{2}d(d-5) + 19 - 2g$ and meet in a subscheme of codimension one, the third component has dimension $\frac{3}{2}d(d-3) + 9 - 2g$. The support of these three components is generically smooth.*

Proof. It suffices to note that the results about $\mathcal{F}_{EX}$ are shown in [12]. □

We believe that the above result remains true if we replace the assumption on the genus by $g \leq \binom{d-3}{2} - 2$. However, proving the statement in this generality seems to require a different approach.

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