ON THE COMPONENTWISE LINEARITY AND THE MINIMAL FREE RESOLUTION OF A TETRAHEDRAL CURVE

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Abstract. A tetrahedral curve is an unmixed, usually non-reduced, one-dimensional subscheme of projective 3-space whose homogeneous ideal is the intersection of powers of the ideals of the six coordinate lines. The second and third authors have shown that these curves have very nice combinatorial properties, and they have made a careful study of the even liaison classes of these curves. We build on this work by showing that they are "almost always" componentwise linear, i.e. their homogeneous ideals have the property that for any d, the degree d component of the ideal generates a new ideal whose minimal free resolution is linear. The one type of exception is clearly spelled out and studied as well. The main technique is a careful study of the way that basic double linkage behaves on tetrahedral curves, and the connection to the tetrahedral curves that are *minimal* in their even liaison classes. With this preparation, we also describe the minimal free resolution of a tetrahedral curve, and in particular we show that in any fixed even liasion class there are only finitely many tetrahedral curves with linear resolution. Finally, we begin the study of the generic initial ideal (gin) of a tetrahedral curve. We produce the gin for arithmetically Cohen-Macaulay tetrahedral curves and for minimal arithmetically Buchsbaum tetrahedral curves, and we show how to obtain it for any non-minimal tetrahedral curve in terms of the gin of the minimal curve in that even liaison class.

Contents

1. Introduction

A *tetrahedral curve* is a curve in $\mathbb{P}^3$ defined by an ideal

$$I = (a,b)^{a_1} \cap (a,c)^{a_2} \cap (a,d)^{a_3} \cap (b,c)^{a_4} \cap (b,d)^{a_5} \cap (c,d)^{a_6} \subset k[a,b,c,d].$$

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These ideals are unmixed of codimension two, and their name comes from the fact that one can view the six lines defined by the ideals of two of the variables as forming the edges of a tetrahedron. In his unpublished Ph.D. thesis [15], Phil Schwartau studied the case in which $a_2 = a_5 = 0$, giving a characterization of when the curves are Cohen-Macaulay in terms of the a_i and describing their minimal free resolutions. Note that when $a_2 = a_5 = 0$, the remaining four lines of support form a complete intersecton of type (2,2).

The general case of a tetrahedral curve, when a_2 and a_5 are not necessarily zero, is studied in [14]. There is a straightforward reduction procedure for tetrahedral curves using basic double linkage. Starting with a tetrahedral curve, one does a sequence of basic double links, getting progressively smaller tetrahedral curves and ending with one of two outcomes. The reduction process could stop with the empty set, which we will call the *trivial curve*, defined by the 6-tuple $(0, 0, 0, 0, 0, 0)$. Alternatively, one might reach a *minimal curve* that cannot be reduced further. An easy numerical test allows one to determine when one has reached a minimal curve, leading to a simple algorithm for the reduction process. Moreover, all the curves in a reduction sequence are in the same even liaison class.

The resolutions of the minimal tetrahedral curves have a particularly nice form. The authors of [14] find their graded Betti numbers explicitly and show that the resolutions are all linear. Additionally, the length of the resolution guarantees that the trivial curve is the only minimal arithmetically Cohen-Macaulay curve. A consequence of the form of the minimal free resolution of minimal tetrahedral curves is that a tetrahedral curve is minimal (in the reduction process) if and only if it is minimal in its even liaison class. As applications, the authors of [14] give a new proof of Schwartau's result characterizing the Cohen-Macaulay curves with $a_2 = a_5 = 0$, classify the 6-tuples of minimal, arithmetically Buchsbaum curves, and explore unobstructedness and the Hilbert scheme of some tetrahedral curves.

In this paper, much of our work is devoted to determining when the ideal of a tetrahedral curve is componentwise linear and the consequences of this characterization. We recall the definition of a componentwise linear ideal.

Definition 1.1. Let I be a homogeneous ideal, and write (I_d) for the ideal generated by the degree d elements of I. We say that I is *componentwise linear* if (I_d) has a linear resolution for all d.

Of course, any ideal with a linear resolution is also componentwise linear. Some other common examples of componentwise linear ideals include strongly stable ideals, squarefree strongly stable ideals, and the **a**-stable ideals of [7].

Componentwise linear ideals were introduced in a paper of Herzog and Hibi [10]. Initially, a primary motivation for studying componentwise linear ideals came from combinatorics and the desire to generalize the notion of having a linear resolution. Eagon and Reiner proved that if Δ is a simplicial complex, and $I_\Delta \subset R = k[x_1, \ldots, x_n]$ is its Stanley-Reisner ideal, then I_Δ has a linear resolution if and only if the Alexander dual Δ^* is Cohen-Macaulay over k [4]. Componentwise linear ideals help extend this statement; I_Δ is componentwise linear if and only if Δ^* is sequentially Cohen-Macaulay, a property that requires a nice filtration on R/I_{Δ^*} in which the quotients are Cohen-Macaulay [10, 11].

In addition, componentwise linear ideals have a number of algebraic properties that make them interesting to study. Herzog and Hibi proved convenient formulas for the graded Betti numbers of a componentwise linear ideal I in terms of the Betti numbers of the (I_d) and $\mathfrak{m}(I_d)$, where $\mathfrak{m}$ is the maximal homogeneous ideal. Moreover, Aramova, Herzog, and Hibi proved that if the characteristic of k is zero, and $\text{gin}(J)$ is the reverse-lex generic initial ideal of J, then J and $\text{gin}(J)$ have the same graded Betti numbers if and only if J is componentwise linear. Thus componentwise linear ideals have the same graded Betti numbers as strongly stable ideals, so there is a lot of structure in their resolutions.

The origin of this work is a confluence of ideas from two places. A remark in [14] notes that there are a number of linear strands in the minimal free resolution of the ideal of a tetrahedral curve. We wanted to find a clear explanation for how these linear strands arise. Additionally, the main result of [6] is that ideals of at most $n+1$ general fat points in $\mathbb{P}^n$ are componentwise linear. One can take these ideals to be the intersection of powers of ideals generated by n of the $n+1$ variables; that is, in $\mathbb{P}^3$, they have the form

$$(b,c,d)^{b_1} \cap (a,c,d)^{b_2} \cap (a,b,d)^{b_3} \cap (a,b,c)^{b_4}.$$

These ideals are similar enough to the ideals of tetrahedral curves that we wondered if one might be able to prove that some large class of tetrahedral curves is componentwise linear, and tests using Macaulay 2 [9] and the MAPLE code from [14] suggested many of our results in the following sections.

The main tool throughout our paper is the reduction process for tetrahedral curves from [14]. We begin our investigation in section 3 by determining in Proposition 3.1 how componentwise linearity persists in a basic double link. This analysis forms the basis for Theorem 4.7 and Corollary 4.8, which characterize which ideals of tetrahedral curves are componentwise linear in terms of the curves to which they reduce. In the case of Schwartau curves, when $a_2 = a_5 = 0$, we can say more, proving in Corollary 4.10 that ideals of Schwartau curves fail to be componentwise linear if and only if $a_1 + a_6 = a_3 + a_4$, and a_1, a_3, a_4, and a_6 are all positive.

As applications of our results on componentwise linearity, we prove a number of statements about the minimal free resolutions of ideals of tetrahedral curves. The ideals J that are not componentwise linear are actually not far from being componentwise linear, which we measure in Proposition 5.2 by comparing the graded Betti numbers of the reverse-lex generic initial ideal $\text{gin}(J)$ to those of J. One consequence is Theorem 5.6, which gives an explicit expression of the regularity of any tetrahedral curve in terms of the a_i. Additionally, Corollary 5.13 describes an easy iterative procedure for calculating the graded Betti numbers of any tetrahedral curve from just the a_i and a knowledge of the graded Betti numbers of the minimal curves from [14].

In section 6, we investigate which tetrahedral curves, in addition to the minimal ones, have linear resolutions. We characterize the arithmetically Cohen-Macaulay curves with linear resolutions in Proposition 6.1 and find all the tetrahedral curves with linear resolutions that are in the even liaison class of two skew lines in Proposition 6.4. In addition, we show in Theorem 6.5 that there are only finitely many tetrahedral curves with a linear resolution in the even liaison class of a tetrahedral curve that is not arithmetically Cohen-Macaulay.

Finally, we conclude with some observations about the reverse-lex generic initial ideal of a tetrahedral curve. The gin is easy to describe in the arithmetically Cohen-Macaulay case, and we discuss how the gin changes with a basic double link in the non-arithmetically Cohen-Macaulay case. In particular, if we know the gin for a minimal non-arithmetically Cohen-Macaulay tetrahedral curve then we know it for any tetrahedral curve in the corresponding even liaison class. We carry out this program for the arithmetically Buchsbaum tetrahedral curves.

Throughout, we will often abuse notation and refer to the ideal $I = (a_1, \dots, a_6)$ or the curve $C = (a_1, \dots, a_6)$ interchangably.

2. Preliminaries

We will denote by R the polynomial ring $k[x_0, x_1, \dots, x_n]$, where k is any field. We also denote by $\mathfrak{m}$ the irrelevant ideal $(x_0, x_1, \dots, x_n)$. Starting with section 4, though, we will follow [15] and [14] and let $R = k[a, b, c, d]$.

Remark 2.1. When we refer to "the smaller curve" in the proofs in this paper, we mean the smaller of the schemes defined by the corresponding ideals, not the smaller of the ideals.

Notation 2.2. For a homogeneous ideal $I \subset R = k[x_1, \dots, x_n]$, we let $I_{\geq d}$ be the ideal generated by all elements of I of degree at least d. Furthermore, I_d will denote the degree d part of I, and (I_d) will denote the ideal generated by the degree d part of I.

We begin with a lemma describing how the graded Betti numbers of I and $I_{\geq d}$ differ.

Lemma 2.3. *Let I be a homogeneous ideal in $S = k[x_1, \dots, x_n]$, and let d be a positive integer. Then for each integer $r \geq 0$ and all i,*

$$\beta_{i,i+d+1+r}(I_{\geq d}) = \beta_{i,i+d+1+r}(I).$$

Proof. We have the short exact sequence

$$0 \to I_{\geq d} \to I \to I/I_{\geq d} \longrightarrow 0.$$

This induces a long exact sequence in Tor: For all $r \geq 0$,

$$\cdots \to \operatorname{Tor}_{i+1}(I/I_{\geq d}, k)_{i+d+1+r} \to \operatorname{Tor}_i(I_{\geq d}, k)_{i+d+1+r} \to \operatorname{Tor}_i(I, k)_{i+d+1+r} \to$$
$$\operatorname{Tor}_i(I/I_{\geq d}, k)_{i+d+1+r} \to \cdots$$

is an exact sequence of k-vector spaces. Moreover, $I/I_{\geq d}$ has finite length; it is zero in degree d and higher and has highest degree socle generator in degree $d-1$. Therefore $I/I_{\geq d}$ has regularity $d-1$, meaning $\beta_{i,i+d+r}(I/I_{\geq d}) = 0$ for all i and all $r \geq 0$.

For $r \geq 0$, because $d + r > d - 1$,

$$\dim_k \operatorname{Tor}_{i+1}(I/I_{\geq d}, k)_{i+d+1+r} = \beta_{i+1,i+d+1+r}(I/I_{\geq d}) = 0.$$

Similarly,

$$\dim_k \operatorname{Tor}_i(I/I_{\geq d}, k)_{i+d+1+r} = \beta_{i,i+d+1+r}(I/I_{\geq d}) = 0.$$

Consequently, as k-vector spaces,

$$\operatorname{Tor}_i(I_{\geq d}, k)_{i+d+1+r} \cong \operatorname{Tor}_i(I, k)_{i+d+1+r}.$$

Hence their dimensions over k are equal, and thus for all $r \geq 0$,

$$\beta_{i,i+d+1+r}(I_{\geq d}) = \beta_{i,i+d+1+r}(I).$$

$\square$

A very basic tool used in [14] and in this paper is that of *basic double linkage*. This very simple but powerful construction was introduced by Lazarsfeld and Rao [12] to describe the even liaison class of a general curve in $\mathbb{P}^3$, but it has seen a wealth of generalizations and applications since then, far too many to list here. We refer the reader to [13] for some of these, although many more have emerged since [13] was published. We recall here the codimension two construction and important facts of basic double linkage, and even this will be a special case (using a linear form instead of a form of any degree) for the purposes needed below. Again, we cite [13] for the proofs. For convenience, in the result below we denote by $\deg(I)$ the degree of the scheme defined by I.

Theorem 2.4. *Let $I \subset R$ be a homogeneous ideal, and let $F \in I$ be a homogeneous polynomial of degree d. Let $L \in R_1$ be a linear form such that L is not a factor of F, i.e. such that (L, F) is a regular sequence. Let J be the ideal $L \cdot I_V + (F)$. J is called a* basic double link *of I. Then*

(a) *We have an exact sequence*

$$0 \to R(-d-1) \to I(-1) \oplus R(-d) \to J \to 0$$

where the first map is given by $C \mapsto (FC, LC)$ and the second is given by $(A, B) \mapsto LA - FB$.

(b) *J is saturated if and only if I is saturated.*

(c) *J is unmixed if and only if I is the saturated ideal of a codimension two subscheme of $\mathbb{P}^n$. In this case we have $\deg(J) = \deg(I) + \deg(F)$.*

We assume from now on that J is unmixed.

(d) *J is linked in two steps to I. Hence basic double linkage preserves the even liaison class of I. In particular, J is arithmetically Cohen-Macaulay if and only if I is arithmetically Cohen-Macaulay. Also, J is locally Cohen-Macaulay and equidimensional if and only if I is locally Cohen-Macaulay and equidimensional.*

3. Basic Double Linkage and Componentwise Linear Ideals

In this section we find initial connections between the construction of basic double linkage in codimension two and componentwise linear ideals. These will be important in the subsequent sections.

Proposition 3.1. *Assume that I is componentwise linear. Let $F \in I$ and let L be a linear form such that (L, F) is a regular sequence. Let $J = L \cdot I + (F)$. Then J is componentwise linear if and only if F is* not *a minimal generator of I.*

Proof. First assume that F is not a minimal generator of I. Let $\deg F = e$. For any d, we have that $J_d = L \cdot I_{d-1} + F \cdot \mathfrak{m}^{d-e}$, where we make the convention that $\mathfrak{m}^{d-e} = 0$ for $d < e$. If $d < e$ then $J_d = L \cdot I_{d-1}$, so (J_d) has a linear resolution since (I_{d-1}) does.

Next we suppose that $d = e$. We have J_e is spanned by $L \cdot I_{e-1}$ and F. It follows that $(J_e) = L \cdot (I_{e-1}) + (F)$. Since F is not a minimal generator of I, $F \in (I_{e-1})$. Hence the ideal (J_e) arises as a basic double link from (I_{e-1}) using L and F. We then have from Theorem 2.4 the exact sequence

$$0 \to R(-e-1) \to (I_{e-1})(-1) \oplus R(-e) \to (J_e) \to 0. \tag{3.1}$$

The mapping cone then gives a linear resolution for (J_e).

Finally, let $d > e$. We know from Theorem 2.4 that we have an exact sequence

$$0 \to R(-e-1) \to I(-1) \oplus R(-e) \to J \to 0. \tag{3.2}$$

We claim that we now have a short exact sequence

$$0 \to \mathfrak{m}^{d-e-1}(-e-1) \to (I_{d-1})(-1) \oplus \mathfrak{m}^{d-e}(-e) \to (J_d) \to 0. \tag{3.3}$$

Indeed, the first map is given by $C \mapsto (FC, LC)$ and the second map is given by $(A, B) \mapsto LA - FB$. Because $F \in I_e$, the kernel of the second map is immediately seen to be isomorphic to $\mathfrak{m}^{d-e-1}$, since (L, F) is a regular sequence and so L and F have no common factor. This gives a diagram

$$\begin{array}{ccccccccccc}
 & & \vdots & & & \vdots & & & & & \\
 & & \downarrow & & & \downarrow & & & & & \\
 & & R(-d-1)^\bullet & & R(-d-1)^\bullet & \oplus & R(-d-1)^\bullet & & & & \\
 & & \downarrow & & & \downarrow & & & & & \\
 & & R(-d)^\bullet & & R(-d)^\bullet & \oplus & R(-d)^\bullet & & & & \\
 & & \downarrow & & & \downarrow & & & & & \\
0 & \to & \mathfrak{m}^{d-e-1}(-e-1) & \to & (I_{d-1})(-1) & \oplus & \mathfrak{m}^{d-e}(-e) & \to & (J_d) & \to & 0 \\
 & & \downarrow & & & \downarrow & & & & & \\
 & & 0 & & & 0 & & & & &
\end{array} \tag{3.4}$$

Since $F\mathfrak{m}^{d-e-1} \subset (I_{d-1})$ we get for all i that every minimal i-th syzygy of $\mathfrak{m}^{d-e-1}$ is an i-th syzygy of (I_{d-1}), thus a minimal syzygy for degree reasons. Hence, the terms in the mapping cone coming from the leftmost column all get split off, leaving a linear resolution for (J_d). This completes one direction of the proof.

Conversely, we assume that F *is* a minimal generator of I, and we show that then J is not componentwise linear. Again suppose $\deg F = e$. We have $J = L \cdot I + (F)$. We will show that (J_e) does not have a linear resolution. Note that J_e is again spanned by $L \cdot I_{e-1}$ and F. Consider the exact sequence

$$0 \to K \to (I_{e-1})(-1) \oplus R(-e) \to (J_e) \to 0$$

where the second map is given by $(A, B) \mapsto LA - FB$, and K is just the kernel. An element of the kernel of this map corresponds to a pair (A, B) for which $LA = FB$. An element of K of degree e corresponds to a pair (A, λ), where $A \in I_{e-1}$, $\lambda \in k$, and $LA = \lambda F$. But $F \notin (I_{e-1})$ since F is a minimal generator, so this is impossible. Hence $K_e = 0$. An element of K of degree $e+1$ corresponds to a pair (A, B) where $A \in (I_{e-1})_e$, B is a linear form, and $LA = FB$. But L and F have no common factor, so up to scalar multiple we have $L = B$ and $A = F$. But F is a minimal generator of I, so again $F \notin (I_{e-1})$. We thus have that also $K_{e+1} = 0$. But this means that K has generators in degree $\geq e+2$. Since by hypothesis (I_{e-1}) has a linear resolution, the mapping cone gives a resolution for (J_e) that cannot be linear. $\square$

Corollary 3.2. *Assume that I is componentwise linear, and let $J = L \cdot I + (F)$, where $F \in I$ and (F, L) is a regular sequence. Assume further that F is a minimal generator of I of degree e. Then (J_d) has a linear resolution if and only if $d \neq e$.*

Proof. We know from Theorem 3.1 that there is at least one d for which (J_d) does not have a linear resolution. We have seen, in fact, that (J_e) does not have a linear resolution, proving one direction here. So we have only to show that if $d \neq e$ then (J_d) has a linear resolution. The proof is very similar to that of Theorem 3.1. If $d < e$ then $J_d = L \cdot I_{d-1}$, and the linearity is clear. If $d > e$ then (3.3) and (3.4) continue to hold, and the linearity of the resolution is proved in the same way. □

Corollary 3.3. *Let $F \in I$ and let L be a linear form such that (L, F) is a regular sequence. Let $J = L \cdot I + (F)$. Assume that F is not a minimal generator of I. Then*

(a) *J is componentwise linear if and only if I is componentwise linear.*
(b) *If I has a linear resolution and generators of degree $e - 1$ then J has a linear resolution if and only if $\deg F = e$.*
(c) *If J has a linear resolution then so does I.*

Proof. Part (a) is a subtle variation of Theorem 3.1 which will, nevertheless, prove useful. If I is componentwise linear then we have already proved the result in Theorem 3.1. So we must assume that J is componentwise linear. The proof is almost identical to that given in Theorem 3.1. We still get the diagram (3.4) if $d > e$, and so we only have to observe that in order for the resulting resolution for J_d to be linear, we must have the resolution for I_{d-1} be linear as well.

If $d = e$, then we argue similarly by using the sequence (3.1).

Parts (b) and (c) follow using similar arguments, using the sequence (3.2). □

4. When is a Tetrahedral Curve Componentwise Linear?

We now apply the results of the preceding section to tetrahedral curves. From now on R will denote the ring $k[a, b, c, d]$. We first recall some basic results from [14].

Proposition 4.1 ([14] Proposition 3.1)**.** *Let $J = (a, b)^{a_1} \cap (a, c)^{a_2} \cap (a, d)^{a_3} \cap (b, c)^{a_4} \cap (b, d)^{a_5} \cap (c, d)^{a_6}$ where not all exponents a_i are zero. Consider the following systems of inequalities:*

$$\begin{array}{rrcl}
(A): & a_1 + a_2 & \geq & a_4, \\
 & a_1 + a_3 & \geq & a_5, \\
 & a_2 + a_3 & \geq & a_6 \\
(C): & a_2 + a_4 & \geq & a_1, \\
 & a_2 + a_6 & \geq & a_3, \\
 & a_4 + a_6 & \geq & a_5
\end{array}
\qquad
\begin{array}{rrcl}
(B): & a_1 + a_4 & \geq & a_2, \\
 & a_1 + a_5 & \geq & a_3, \\
 & a_4 + a_5 & \geq & a_6 \\
(D): & a_3 + a_5 & \geq & a_1, \\
 & a_3 + a_6 & \geq & a_2, \\
 & a_5 + a_6 & \geq & a_4.
\end{array}$$

For $1 \leq i \leq 6$ let $a_i' = \max\{0, a_i - 1\}$. Then we have

(i) *$(A) \Leftrightarrow J$ is a basic double link of*

$$(a, b)^{a_1'} \cap (a, c)^{a_2'} \cap (a, d)^{a_3'} \cap (b, c)^{a_4} \cap (b, d)^{a_5} \cap (c, d)^{a_6}$$

using $F = b^{a_1} c^{a_2} d^{a_3}$ and $G = a$.

(ii) $(B) \Leftrightarrow J$ *is a basic double link of*

$$(a,b)^{a_1'} \cap (a,c)^{a_2} \cap (a,d)^{a_3} \cap (b,c)^{a_4'} \cap (b,d)^{a_5'} \cap (c,d)^{a_6}.$$

using $F = a^{a_1}c^{a_4}d^{a_5}$ *and* $G = b$.

(iii) $(C) \Leftrightarrow J$ *is a basic double link of*

$$(a,b)^{a_1} \cap (a,c)^{a_2'} \cap (a,d)^{a_3} \cap (b,c)^{a_4'} \cap (b,d)^{a_5} \cap (c,d)^{a_6'}.$$

using $F = a^{a_2}b^{a_4}d^{a_6}$ *and* $G = c$.

(iv) $(D) \Leftrightarrow J$ *is a basic double link of*

$$(a,b)^{a_1} \cap (a,c)^{a_2} \cap (a,d)^{a_3'} \cap (b,c)^{a_4} \cap (b,d)^{a_5'} \cap (c,d)^{a_6'}.$$

using $F = a^{a_3}b^{a_5}c^{a_6}$ *and* $G = d$.

Definition 4.2 ([14] Theorem 5.1)**.** A non arithmetically Cohen-Macaulay tetrahedral curve C is *minimal* if either of the following equivalent conditions holds:

(a) The ideal I_C does not admit any reduction of the type given in parts (A) to (D) of Proposition 4.1;
(b) C is minimal in its even liaison class (cf. [13]).

Corollary 4.3 ([14] Corollary 3.5)**.** *Consider a tetrahedral curve* $C = (a_1, a_2, a_3, a_4, a_5, a_6)$ *where not all* a_i *are 0. Assume without loss of generality that* $a_6 = \max\{a_1, \ldots, a_6\}$*. Then* C *is minimal if and only if*

$$\begin{array}{rcl} a_1 & > & \max\{a_3 + a_5, a_2 + a_4\} \text{ and} \\ a_6 & > & \max\{a_4 + a_5, a_2 + a_3\} \end{array}$$

Theorem 4.4 ([14] Theorem 4.2)**.** *Every non-trivial* minimal *tetrahedral curve has a linear minimal free resolution.*

More precisely, if the curve C *is defined by* $(a_1, a_2, a_3, a_4, a_5, a_6)$ *and* $a_6 = \max\{a_i\} > 0$ *then its minimal free resolution has the form*

$$0 \to R^{\beta_3}(-a_1 - a_6 - 2) \to R^{\beta_2}(-a_1 - a_6 - 1) \to R^{\beta_1}(-a_1 - a_6) \to I_C \to 0$$

where

$$\begin{aligned} \beta_1 &= (a_1+1)(a_6+1) - \sum_{i=2}^{5} \frac{a_i(a_i+1)}{2} \\ \beta_2 &= 2a_1a_6 + a_1 + a_6 - \sum_{i=2}^{5} a_i(a_i+1) \\ \beta_3 &= a_1a_6 - \sum_{i=2}^{5} \frac{a_i(a_i+1)}{2}. \end{aligned}$$

In order to have an (almost) canonical way to reduce to a minimal tetrahedral curve, we use facets of maximal weight. Recall that the *weight of a facet* is the sum of the weights of the edges forming its boundary.

Lemma 4.5 ([14] Lemma 3.8)**.** *Let* $C = (a_1, a_2, a_3, a_4, a_5, a_6)$ *be a non-trivial tetrahedral curve. If* C *is not minimal then one can reduce any of its facets of maximal weight.*

Example 4.6. Consider the curve $(3,3,3,1,2,4)$. The facets have the following weights: $a_1+a_2+a_3=9$, $a_1+a_4+a_5=6$, $a_2+a_4+a_6=8$ and $a_3+a_5+a_6=9$. For maximal weight there is a tie between the first and the last, and either reduction (i.e. using (A) or (D) in Proposition 4.1) is possible. Note that it is also possible to reduce using (C) (but not (B)), but the algorithm that we will use in this paper restricts to facets of maximal weight, so we do not use this option. The following follows all possible reductions using facets of maximal weight:

$$\begin{tikzcd}
& (3,3,3,1,2,4) \arrow[dl, "(A)"'] \arrow[dr, "(D)"] & \\
(2,2,2,1,2,4) \arrow[dr, "(D)"'] & & (3,3,2,1,1,3) \arrow[dl, "(A)"] \\
& (2,2,1,1,1,3) \arrow[d, "(C)"] & \\
& (2,1,1,0,1,2) \arrow[dl, "(A)"'] \arrow[dr, "(D)"] & \\
(1,0,0,0,1,2) \arrow[dr, "(D)"'] & & (2,1,0,0,0,1) \arrow[dl, "(A)"] \\
& (1,0,0,0,0,1) &
\end{tikzcd}$$

We now begin the study of which tetrahedral curves are componentwise linear.

Theorem 4.7. *Let $J=(a_1,a_2,a_3,a_4,a_5,a_6)$ be a non-trivial tetrahedral curve.*

(a) *If J is minimal then it has a linear resolution, and hence is componentwise linear. So from now on we assume that J is not minimal.*
(b) *If (up to permutation of the variables) $J=(0,r,r,r,r,0)$ (i.e. if J has equal non-trivial weights and is supported on a complete intersection of type $(2,2)$) then J has a pure resolution that is not linear, and hence is not componentwise linear.*
(c) *Suppose that J reduces to another tetrahedral ideal I following the algorithm of* [14], *i.e. using a facet of maximal weight, and using one of the reductions (A), (B), (C) or (D) of Proposition 4.1. (If J is not minimal, it is always possible to reduce using a facet of maximal weight, thanks to Lemma 4.5.) Then the polynomial F that is prescribed by that algorithm is a minimal generator of I if and only if J is of the form described in (b).*

Proof. Statement (a) follows from [14] Theorem 4.2, which in particular shows that J has a linear resolution. For (b), we have that J is the r-th power of a complete intersection $(A,B)=(ab,cd)$ of type (2,2). In fact, it is easy to see that $(ab,cd)^r\subset J$, hence we get equality because both ideals have the same degree. Now the result follows from the fact that its Hilbert-Burch matrix is

$$\begin{bmatrix} A & B & 0 & 0 & 0 & & 0 & 0 \\ 0 & A & B & 0 & 0 & & 0 & 0 \\ 0 & 0 & A & B & 0 & & 0 & 0 \\ & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & 0 & & A & B \end{bmatrix}$$

where there are r rows and $r+1$ columns, and all entries have degree 2.

For (c), without loss of generality assume that a_1, a_2, a_3 give the facet of maximal weight, so that we use the reduction (A). We then have $I = (a_1', a_2', a_3', a_4, a_5, a_6)$ and $F = b^{a_1} c^{a_2} d^{a_3}$. Suppose first that F is a minimal generator of I. We will show that then it must be of the type described in (b).

The fact that F is a minimal generator of I means that if we reduce any of the exponents of F, the result is no longer in I. Since $F = b^{a_1} c^{a_2} d^{a_3}$, it is clear that F vanishes on the components $(a,b)^{a_1'}$, $(a,c)^{a_2'}$ and $(a,d)^{a_3'}$. The condition that F vanishes on $(b,c)^{a_4}$ is given by the inequality $a_1 + a_2 \geq a_4$. Similarly, the condition that F vanishes on $(b,d)^{a_5}$ is given by the inequality $a_1 + a_3 \geq a_5$ and the condition that F vanishes on $(c,d)^{a_6}$ is given by the inequality $a_2 + a_3 \geq a_6$. To say that reducing any one of the exponents of F by one makes the result no longer be in I means that *two* of these inequalities must in fact be equalities. Indeed, this is easily seen if a_1, a_2, a_3 are all positive. Assume that without loss of generality $a_1 = 0$ and only one of these inequalities is an equality. Then, this must be $a_2 + a_3 = a_6$. And by the assumption for the time being we have $a_2 > a_4$ and $a_3 > a_5$. But $a_1 + a_2 + a_3$ is the largest weight of a facet, thus in particular, $a_6 = a_2 + a_3 \geq a_3 + a_5 + a_6$, hence we get $a_3 = a_5 = 0$, a contradiction.

Therefore, we may assume without loss of generality that

$$J = (a_1, a_2, a_3, a_1 + a_2, a_1 + a_3, a_6).$$

We assumed that a_1, a_2, a_3 give the facet of maximal weight for J. This means, in particular, that a_1, a_4, a_5 do not give a facet of greater weight, i.e.

$$a_1 + (a_1 + a_2) + (a_1 + a_3) \leq a_1 + a_2 + a_3.$$

This forces $a_1 = 0$, and $I = (0, a_2, a_3, a_2, a_3, a_6)$. Similarly we have that a_2, a_4, a_6 do not give a facet of greater weight for J, and a_3, a_5, a_6 do not give a facet of greater weight, so

$$\left. \begin{array}{rcl} a_2 + a_2 + a_6 & \leq & a_2 + a_3 \\ a_3 + a_3 + a_6 & \leq & a_2 + a_3 \end{array} \right\} \Rightarrow \begin{array}{rcl} a_2 + a_6 & \leq & a_3 \\ a_3 + a_6 & \leq & a_2 \end{array}$$

which means

$$a_2 + 2a_6 \leq a_3 + a_6 \leq a_2,$$

so also $a_6 = 0$. Hence in fact J is of the form $(0, a_2, a_3, a_2, a_3, 0)$. But then the two inequalities above give $a_2 \leq a_3$ and $a_3 \leq a_2$, which means that $a_2 = a_3$. So we have shown that if we do the reduction via Proposition 3.1 of [14] and if the resulting F is a minimal generator of I, then the ideal J that we started with must be of the type described in (b), i.e. must be supported on a complete intersection of type (2,2), with equal weights on each component. (In particular, I and J must be arithmetically Cohen-Macaulay.)

Conversely, suppose that J is reduced to I and J is of the form described in (b). Without loss of generality say that $J = (0, r, r, r, r, 0)$. Without loss of generality suppose that we are using reduction (A). Then I is $(0, r-1, r-1, r, r, 0)$ and consequently F is $c^r d^r$. It is clear that F is a minimal generator of I, since $c^{r-1}d^r$ is not in $(b, c)^r$ and $c^r d^{r-1}$ is not in $(b, d)^r$. □

Corollary 4.8. *Let $J = (a_1, \dots, a_6)$ be a non-trivial tetrahedral curve.*

(a) *If J is not arithmetically Cohen-Macaulay then J is componentwise linear.*
(b) *Assume that J is arithmetically Cohen-Macaulay. We can reduce J to the trivial curve in a finite sequence of steps, each time using a facet of maximal weight and applying* [14] *Proposition 3.1. Then the following are equivalent:*
  (i) *J is componentwise linear;*
  (ii) *this sequence of steps does not include any curve of the type described in Theorem 4.7 (b);*
  (iii) *this sequence of steps does not include a complete intersection of type (2,2), i.e. does not include any of the curves $(0, 1, 1, 1, 1, 0)$, $(1, 0, 1, 1, 0, 1)$, or $(1, 1, 0, 0, 1, 1)$.*

Proof. Assume that J is not arithmetically Cohen-Macaulay. If it is minimal then by Theorem 4.7 (a) it is componentwise linear. If it is not minimal then we can reduce via facets of maximal weight to a minimal curve. In each step, the polynomial F used is not a minimal generator of the smaller curve I, thanks to Theorem 4.7 (c) and the fact that J is not arithmetically Cohen-Macaulay. Then the statement of (a) follows from Theorem 3.1 and induction on the number of steps to a minimal curve.

For (b), the fact that (ii) implies (iii) is trivial, and the implication (iii) $\Rightarrow$ (ii) follows since a curve of type $(0, r, r, r, r, 0)$ reduces to one of type $(0, 1, 1, 1, 1, 0)$.

Assume that (ii) holds. Then from Theorem 4.7 (c), each step of the procedure of reducing by maximal facets involves a polynomial F that is not a minimal generator of the smaller curve. Hence by Corollary 3.3, each J is componentwise linear if and only if the next curve I is componentwise linear. But one can easily check that in reducing to the trivial curve via facets of maximal weight, eventually one passes through a curve consisting of all 0's and 1's. By hypothesis we do not pass through a complete intersection of type (2,2) (i.e. the curve $(0, 1, 1, 1, 1, 0)$, up to permutation). One can easily check that all other tetrahedral curves with only entries that are 0 or 1 are componentwise linear. Hence by induction the tetrahedral curve J that we started with is componentwise linear.

Conversely, assume that J is arithmetically Cohen-Macaulay and componentwise linear. Again we reduce by facets of maximal weight down to the trivial curve. Suppose that at some step we reach a curve of the type described in Theorem 4.7 (b), and consider the first such instance. We have seen in Theorem 4.7 (c) that the form F used in the reduction is a minimal generator if and only if the larger curve (corresponding to J) is of the form described in Theorem 4.7 (b). In our situation we have arrived at the first such curve, so the larger curve in each step has not been of this form. Hence each step in this process has used a form F that was not a minimal generator of the smaller curve. Hence by Corollary 3.3, since we started with an ideal J that was componentwise linear, each of the smaller curves had ideals I that are also componentwise linear. But reaching a curve of the type in Theorem 4.7 (b) we have obtained one that is not componentwise linear. This contradiction completes the proof. □

Definition 4.9. A *Schwartau curve* is a tetrahedral curve $C = (a_1, a_2, a_3, a_4, a_5, a_6)$ for which $a_2 = a_5 = 0$.

Note that a Schwartau curve is supported on a complete intersection of type (2,2). These curves were studied by P. Schwartau in his thesis [15].

Corollary 4.10. *Let J be the ideal of a Schwartau curve. Then J fails to be componentwise linear if and only if all of a_1, a_3, a_4, a_6 are > 0 and $a_1 + a_6 = a_3 + a_4$.*

Proof. We reduce J to the trivial curve by a sequence of steps using the reduction of [14], Proposition 3.1, and using facets of maximal weight. By Corollary 4.8, J fails to be componentwise linear if and only if this reduction includes the curve $(1, 0, 1, 1, 0, 1)$ (this time it must be precisely this curve, not up to permutation).

Suppose that J fails to be componentwise linear. If any of the a_i are 0, then clearly we cannot hope to reach $(1, 0, 1, 1, 0, 1)$. But note that each step in the reduction reduces both sums $a_1 + a_6$ and $a_3 + a_4$ by 1, so if these sums are not equal to begin with, they will never be equal. Hence we will never reach $(1, 0, 1, 1, 0, 1)$. Hence we must have the claimed equality.

Conversely, assume that all $a_i > 0$ and that $a_1 + a_6 = a_3 + a_4$. The maximal facet will always include $\max\{a_1, a_6\}$ and $\max\{a_3, a_4\}$ (and the third edge is 0). Hence since $a_1 + a_6 = a_3 + a_4$, we eventually arrive at $(1, 0, 1, 1, 0, 1)$, so J is not componentwise linear. □

Note that the curves considered in Corollary 4.10 are automatically arithmetically Cohen-Macaulay, thanks to Corollary 4.8 (a).

Corollary 4.11. *Let $(a_1, a_2, a_3, a_4, a_5, a_6)$ be a tetrahedral curve C. Consider the sums $a_1 + a_6, a_2 + a_5, a_3 + a_4$. If the curve fails to be componentwise linear, then the two larger of these sums are equal.*

Proof. We know that if C is not componentwise linear, then it is arithmetically Cohen-Macaulay and reduces via facets of maximal weight to one of the curves listed in Corollary 4.8 (b)(iii). Notice that for any of these curves the two larger sums equal two and the third is zero. But each basic double link increases the two larger sums by one and the third sum by zero or one. Hence, the claim for C follows. □

The converse to this statement is not true. Here is a counterexample.

Example 4.12. Let $J = (10, 1, 2, 3, 10, 1)$. Then this curve is arithmetically Cohen-Macaulay, but is componentwise linear because the reduction to the trivial curve does not pass through any of the curves listed in Corollary 4.8 (b).

5. The Minimal Free Resolution of a Tetrahedral Curve

In this section we describe the whole minimal free resolution of a tetrahedral curve. In particular, we make observations about the minimal generators and about the regularity.

We first prove a lemma about the degree of the monomial F used in the algorithm from [14] for reducing a tetrahedral curve. We focus on the case in which our ideals are not componentwise linear.

Lemma 5.1. *Let J be the ideal of a tetrahedral curve $(a_1, \ldots, a_6)$ that is not of type $(0, r, r, r, r, 0)$, even with the variables permuted. Assume that J reduces using the algorithm from [14] to a curve of type $(0, r, r, r, r, 0)$ (possibly with the variables permuted), and hence is not componentwise linear. Assume further that J has its lowest degree minimal generators in degree p. If I is the ideal of the curve obtained by reducing the facet of maximal weight of the curve of J using the algorithm from [14], and $J = L \cdot I + (F)$, then $\deg F \geq p + 1$.*

Proof. Note that I has its lowest degree minimal generators in degree $p-1$ because F is not a minimal generator of I. Therefore F has degree at least p, and we wish to show that it has degree at least $p+1$. Suppose to the contrary that $\deg F = p$ and that $F = b^{a_1}c^{a_2}d^{a_3}$. Then F is b, c, or d times a minimal generator of I; without loss of generality, say $b^{a_1-1}c^{a_2}d^{a_3}$ is a minimal generator of I. Then $b^{a_1-1}c^{a_2-1}d^{a_3} \notin I$, so $a_1 - 1 + a_2 - 1 < a_4$ or $a_2 - 1 + a_3 < a_6$. Similarly, $b^{a_1-1}c^{a_2}d^{a_3-1} \notin I$, and thus $a_1 - 1 + a_3 - 1 < a_5$ or $a_2 - 1 + a_3 < a_6$.

Suppose first that $a_2 + a_3 - 1 < a_6$; then $a_2 + a_3 = a_6$ because $b^{a_1-1}c^{a_2}d^{a_3} \in I$. We are assuming that $a_1 + a_2 + a_3$ is the maximal weight of a facet, and therefore $a_1 + a_6 = a_1 + a_2 + a_3 \geq a_2 + a_4 + a_6$. Consequently, $a_1 \geq a_2 + a_4$. Also, we have $a_1 \geq a_4 - a_2$ because $b^{a_1}c^{a_2}d^{a_3} \in I$, and hence $a_1 \geq a_4$. Similarly, using that $a_1 + a_2 + a_3 \geq a_3 + a_5 + a_6$, we conclude that $a_1 \geq a_5$. As a result,

$$a_1 + a_6 = a_1 + a_2 + a_3 \geq a_5 + a_2 + a_3 \quad \text{and} \quad a_1 + a_6 \geq a_4 + a_2 + a_3.$$

This says that $a_1 + a_6$ is equal to the maximum of $\{a_1 + a_6, a_2 + a_5, a_3 + a_4\}$.

Because J reduces to a curve of type $(0, r, r, r, r, 0)$, it is not componentwise linear. Hence by Corollary 4.11, $a_1 + a_6$ is equal to either $a_2 + a_5$ or $a_3 + a_4$; without loss of generality, assume it is $a_2 + a_5$. We have

$$a_1 + a_2 + a_3 = a_1 + a_6 = a_2 + a_5,$$

so $a_1 + a_3 = a_5$. Since $a_1 \geq a_5$, this forces $a_3 = 0$. Therefore $a_2 = a_6$, and because $a_1 + a_6 = a_2 + a_5$, $a_1 = a_5$. Hence J is the ideal of a curve $(a_1, a_2, 0, a_4, a_1, a_2)$. But then I is the ideal of a curve of the form $(a_1 - 1, a_2', 0, a_4, a_1, a_2)$. We know that $b^{a_1-1}c^{a_2}d^{a_3} = b^{a_1-1}c^{a_2}$ is a minimal generator of I, but

$$b^{a_1-1}c^{a_2} \notin (b, d)^{a_1},$$

a contradiction.

As a result, we conclude that $a_1 + a_3 - 1 = a_5$ and $a_1 + a_2 - 1 = a_4$. Therefore

$$a_2 + a_5 = (a_4 - a_1 + 1) + a_5 = (a_4 - a_1 + 1) + (a_1 + a_3 - 1) = a_3 + a_4.$$

Because J is not componentwise linear, $a_2 + a_5 = a_3 + a_4$ must be the maximum value among $a_1 + a_6$, $a_2 + a_5$, and $a_3 + a_4$ since by Corollary 4.11, the largest two of those are equal. Using this and the fact that $a_1 + a_3 = a_5 + 1$, we have

$$a_3 + a_4 + 1 = a_2 + a_5 + 1 = a_1 + a_2 + a_3 \geq a_1 + a_4 + a_5,$$

where the inequality holds because $a_1+a_2+a_3$ gives the maximal weight of a facet. Hence

$$a_3+1 \geq a_1+a_5 = a_1+(a_1+a_3-1),$$

so $2 \geq 2a_1$, and $a_1 \leq 1$. But $a_1 \neq 0$ because if it were zero, $b^{a_1-1}c^{a_2}d^{a_3}$ would not be a minimal generator of I. Therefore $a_1 = 1$, which implies that $a_3 = a_5$ and $a_2 = a_4$. Thus J is the ideal of a curve $(1, a_2, a_3, a_2, a_3, a_6)$, and $F = bc^{a_2}d^{a_3}$. Note that $a_2 \neq 0 \neq a_3$, for if one of them were zero, then J could not reduce to a curve of the form $(0, r, r, r, r, 0)$, even with the variables permuted. Consequently, I is the ideal of a curve $(0, a_2-1, a_3-1, a_2, a_3, a_6)$.

Next, we claim that $a_6 \leq 1$. To see this, note that $1+a_2+a_3$ gives the maximal weight of a facet of J. Therefore

$$1+a_2+a_3 \geq 2a_2+a_6 \quad \text{and} \quad 1+a_2+a_3 \geq 2a_3+a_6;$$

adding these inequalities gives the claim.

We wish to show that $m = bc^{a_2-1}d^{a_3-1} \in I$. If so, then it has degree at least $p-1$, which implies that $\deg F = \deg bc^{a_2}d^{a_3} \geq p+1$. If $a_2+a_3-2 \geq a_6$, we can conclude that $m \in I$. This inequality holds if $a_6 = 0$ since a_2 and a_3 are both at least one. If $a_2+a_3-2 < a_6 = 1$, then $a_2 = a_3 = 1$, and I is the ideal of the curve $(0, 0, 0, 1, 1, 1)$, which does not reduce to an ideal of the form $(0, r, r, r, r, 0)$. Consequently, $\deg F \geq p+1$. $\square$

The lemma allows us to compare the resolutions of J and $\operatorname{gin}(J)$ for any ideal J of a tetrahedral curve.

Proposition 5.2. *Let J be the ideal of a nontrivial tetrahedral curve.*

(a) *If J is componentwise linear, then the graded Betti numbers of J and* $\operatorname{gin}(J)$ *are the same.*
(b) *Suppose J is not componentwise linear and has its lowest degree minimal generators in degree p. Assume that J reduces using the algorithm from* [14] *to a curve of type* $(0, r, r, r, r, 0)$ *(possibly with the variables permuted), with $r > 0$, but not* $(0, r+1, r+1, r+1, r+1, 0)$. *Then the graded Betti numbers of* $\operatorname{gin}(J)$ *and J are the same except that* $\operatorname{gin}(J)$ *has r additional minimal generators and syzygies in degree $p+1$.*

Proof. Part (a) is immediate from Theorem 1.1 in [1]. For part (b), note that J has projective dimension two because R/J is Cohen-Macaulay. Suppose first that J is the ideal of a $(0, r, r, r, r, 0)$ curve. Then it is easy to compute (see Theorem 4.7(b)) that J has resolution

$$0 \to R(-2r-2)^r \to R(-2r)^{r+1} \to J \to 0.$$

Since the regularity, Hilbert functions, and projective dimensions of J and $\operatorname{gin}(J)$ are the same, the only possible differences in their Betti numbers are additional generators and syzygies of $\operatorname{gin}(J)$ in degree $2r+1$. Because $\operatorname{gin}(J)$ is strongly stable, it is componentwise linear, and thus $(\operatorname{gin}(J)_{2r})$ has a linear resolution. Therefore it must have r minimal syzygies of degree $2r+1$ on the $r+1$ minimal generators of degree $2r$. Thus there are r additional generators of degree $2r+1$ to preserve the Hilbert function. Note that (J_{2r+1+s}) has a linear resolution for all $s \geq 0$ since the regularity of J is $2r+1$.

Suppose now that J, the ideal of a curve $(a_1, \ldots, a_6)$, is a basic double link of I, so that $J = L \cdot I + (F)$, where L is a linear form. We assume that I is obtained from J by reducing a facet of maximal weight. Suppose further that J is not a curve of type $(0, r, r, r, r, 0)$ but reduces to a curve of that form (again possibly with the variables permuted) and not $(0, r+1, r+1, r+1, r+1, 0)$. We have that I has its lowest degree minimal generators in degree $p-1$ because F is not a minimal generator of I. For the induction hypothesis, we assume that I has $r+1$ minimal generators of degree $p-1$, no minimal syzygies of degree p, and that (I_{p+s}) has a linear resolution for all $s \geq 0$.

By Lemma 5.1, $\deg F \geq p+1$. Therefore J has $r+1$ minimal generators of lowest degree p and no syzygies of degree $p+1$. Because F is not a minimal generator of I, it follows from the same arguments as in Theorem 3.1 and Corollary 3.2 that since (I_{p+s}) has a linear resolution for all $s \geq 0$, (J_{p+1+s}) does also. That means that $J_{\geq(p+1)}$ is componentwise linear, where $J_{\geq(p+1)}$ is the ideal generated by all elements of J with degree at least $p+1$.

By Lemma 2.3, $\beta_{i,i+j}(J_{\geq(p+1)}) = \beta_{i,i+j}(J)$ for all $j \geq p+2$. Consequently, $\beta_{i,i+j}(J) = \beta_{i,i+j}(\text{gin}(J))$ for all $j \geq p+2$ (and for all $j < p$ since those Betti numbers are zero). Because the gin preserves the Hilbert function, we have $\beta_{0,p}(J) = \beta_{0,p}(\text{gin}(J))$, and also $\beta_{1,p+2}(J) = \beta_{1,p+2}(\text{gin}(J))$ since the number of generators of degree $p+2$ is the same for both ideals. As a result, the only possible changes are in degree $p+1$. But $\text{gin}(J)$ is strongly stable, and thus since J has $r+1$ minimal generators of degree p, $\text{gin}(J)$ must have r syzygies of degree $p+1$. Consequently, $\text{gin}(J)$ has r additional minimal generators of degree $p+1$ to preserve the Hilbert function. □

Example 5.3. Let J be the ideal of the tetrahedral curve $(2, 5, 5, 5, 5, 0)$. Then J reduces to the curve $(0, 4, 4, 4, 4, 0)$, and J has resolution

$$0 \to R(-13)2 \oplus R(-12)4 \to R(-12)2 \oplus R(-10)5 \to J \to 0.$$

The gin of J must add four minimal generators and syzygies of degree 11, and it has resolution

$$0 \to R(-13)2 \oplus R(-12)4 \oplus R(-11)4 \to R(-12)2 \oplus R(-11)4 \oplus R(-10)5 \to \text{gin}(J) \to 0.$$

As a consequence, we get a description of the regularity of the ideal of any tetrahedral curve in terms of the degree of the highest degree minimal generator.

Corollary 5.4. *Let J be the ideal of a nontrivial tetrahedral curve.*

(a) *If J is the ideal of a curve of the form $(0, r, r, r, r, 0)$, then the regularity of J is $2r+1$.*

(b) *If J is not the ideal of a curve of the form $(0, r, r, r, r, 0)$ (possibly with the variables permuted), then the regularity of J is the degree of the largest degree minimal generator of J.*

Proof. Part (a) is immediate from the resolution of an ideal of a curve of that form. Part (b) is clear when J is componentwise linear, so suppose that J is not componentwise linear. Note that if p is the smallest degree in which J has minimal generators, then J also has generators in a higher degree $p+s$ by Lemma 5.1. Because $J_{\geq p+1}$ is componentwise linear, the regularity of $J_{\geq p+1}$ is equal to the highest degree in which it has a minimal generator, and these invariants are the same as for J. □

We can use Corollary 5.4 to get a more precise statement about the regularity of a tetrahedral curve that can be read directly from the 6-tuple $(a_1, \ldots, a_6)$. First, we prove a lemma that will serve as the inductive step in our next result.

Lemma 5.5. *Let J be the ideal of a nonminimal tetrahedral curve. Suppose $J = L \cdot I + (F)$ is a basic double link of I, where I is obtained from J by reducing a facet of maximal weight. Assume also that the maximal degree of a minimal generator of I is equal to the maximal weight of a facet of I. Then F is a minimal generator of I, and the highest degree in which J has a minimal generator is equal to the maximal weight of a facet of J.*

Proof. Note first that the ideals corresponding to $(0, r, r, r, r, 0)$ have all their minimal generators in degree $2r$, and thus the result holds if J is of that form. So suppose that J corresponds to the curve $(a_1, a_2, a_3, a_4, a_5, a_6)$, which is not of the form $(0, r, r, r, r, 0)$, and $a_1 + a_2 + a_3$ gives the maximal weight of a facet of this curve. Then the curve corresponding to I has the form $(a'_1, a'_2, a'_3, a_4, a_5, a_6)$, where $a'_i = \max\{0, a_i - 1\}$. We have $J = L \cdot I + (F)$, where $\deg F = a_1 + a_2 + a_3$. Thus we need to show that $a_1 + a_2 + a_3$ is at least as large as the maximal weight of a facet of the curve corresponding to I plus one (adding one because L is a linear form).

If the maximal weight of a facet of I is $a'_1 + a'_2 + a'_3$, then it is clear that $a_1 + a_2 + a_3 \geq a'_1 + a'_2 + a'_3 + 1$, so $\deg F$ is a minimal generator of J of highest degree. Suppose the maximal weight of a facet for I is $a'_1 + a_4 + a_5$; the other two remaining cases are the same. Suppose $a_1 + a_2 + a_3 < a'_1 + a_4 + a_5 + 1$. Then

$$a_1 + a_2 + a_3 \leq a'_1 + a_4 + a_5 \leq a_1 + a_4 + a_5.$$

Because $a_1 + a_2 + a_3 \geq a_1 + a_4 + a_5$, all the inequalities are equalities, which means that $a'_1 = a_1 = 0$ and $a_2 + a_3 = a_4 + a_5$. Therefore $a_1 + a_4 + a_5$ also gives the maximal weight of a facet of J, and hence

$$a_4 + a_5 \geq a_2 + a_4 + a_6 \quad \text{and} \quad a_4 + a_5 \geq a_3 + a_5 + a_6.$$

Adding these inequalities together and using the fact that $a_2 + a_3 = a_4 + a_5$, we have $2a_6 \leq 0$, so $a_6 = 0$. Thus J has the form $(0, a_2, a_3, a_4, a_5, 0)$.

Because $0 + a_2 + a_3$ gives the maximal weight of a facet of J, we have

$$a_2 + a_3 \geq a_3 + a_5 \quad \text{and} \quad a_2 + a_3 \geq a_2 + a_4,$$

so $a_2 \geq a_5$ and $a_3 \geq a_4$. But $a_2 + a_3 = a_4 + a_5$, so $a_2 = a_5$ and $a_3 = a_4$, and J has the form $(0, a_2, a_3, a_3, a_2, 0)$.

If $a_2 = a_3$, then J is of the form $(0, r, r, r, r, 0)$, contradicting our assumption that it was not. Otherwise, by Corollary 4.3, J is minimal, which is again a contradiction. $\square$

We can now express the regularity of any tetrahedral curve explicitly.

Theorem 5.6. *Let J be the ideal of a nontrivial tetrahedral curve.*

(a) *If J is the ideal of a curve of the form $(0, r, r, r, r, 0)$, possibly with the variables permuted, then the regularity of J is $2r + 1$.*
(b) *If J is the ideal of a minimal curve $(a_1, \ldots, a_6)$, assume without loss of generality that a_6 is the largest of the a_i. Then the regularity of J is $a_1 + a_6$, which is strictly greater than the weight of a maximal facet.*
(c) *Suppose J is the ideal of a curve $(a_1, \ldots, a_6)$ that is not minimal and not of the form in (a). Then the regularity of J is the maximal weight of a facet.*

Proof. Part (a) follows from computing the resolution of ideals of this type; see Theorem 4.7(b). Consider now part (b). The value of the regularity is an immediate consequence of Theorem 4.4. For the inequality, assume without loss of generality that $a_1 + a_2 + a_3$ gives the maximal weight of a facet; the argument is similar if a_6 is included in the maximal weight of a facet. By Corollary 4.3, $a_6 > a_2 + a_3$, and thus $a_1 + a_6 > a_1 + a_2 + a_3$.

We turn now to part (c). First, we consider the case in which J is arithmetically Cohen-Macaulay but not of the form $(0, r, r, r, r, 0)$, even with the variables permuted. It is easy to check that all curves of weight at most three and all a_i zero or one have regularity equal to the maximal weight of a facet; also, the regularity is equal to the highest degree of a minimal generator in each case. Moreover, all arithmetically Cohen-Macaulay J reduce to these cases. We proceed by induction with these curves as the base case.

We reduce J down to a base case with the algorithm from [14], always reducing by a facet of maximal weight. For the induction hypothesis, assume the following: All ideals M below J in the reduction from J down to a base case have the property that the highest degree of a minimal generator of M is equal to the maximal weight of a facet of M. Suppose that I is obtained by reducing a facet of J of maximal weight. By the induction hypothesis, the maximal degree of a minimal generator of I is the maximal weight of a facet of I. By Lemma 5.5, this implies that the maximal degree of a minimal generator of J is the maximal weight of a facet of J. We conclude from Corollary 5.4 that this quantity is equal to the regularity of J.

Finally, we consider the case in which J is not arithmetically Cohen-Macaulay. Suppose first that I is the ideal of a minimal, not arithmetically Cohen-Macaulay curve, and suppose I is obtained from J by reducing a facet of maximal weight as in the algorithm from [14]. Then $J = L \cdot I + (F)$, where L is a linear form. Without loss of generality, suppose that J is the ideal of a curve $(a_1, \ldots, a_6)$ with $\deg F = a_1 + a_2 + a_3$ giving the maximal weight of a facet. Then I is the ideal of a curve $(a_1', a_2', a_3', a_4, a_5, a_6)$. We may assume that $a_6 = \max\{a_1', a_2', a_3', a_4, a_5, a_6\}$; the argument is similar if the maximal weight of a facet of I includes a_6.

The regularity of I is $a_1' + a_6$, and thus the minimal generators of $L \cdot I$ have degree $a_1' + a_6 + 1$. Hence $\operatorname{reg}(J) \geq a_1' + a_6 + 1$. By Corollary 5.4, the regularity of J is the maximal degree of a minimal generator. Therefore

$$\operatorname{reg}(J) = \max\{\deg F = a_1 + a_2 + a_3, a_1' + a_6 + 1\}.$$

We want to show that $a_1 + a_2 + a_3 \geq a_1' + a_6 + 1$.

Initially, note that $a_1' = a_1 - 1$; otherwise, $a_1' = 0$, contradicting Corollary 4.3. Thus we need to show that $a_2 + a_3 \geq a_6$. Suppose instead that $a_2 + a_3 < a_6$. We show that J is then the ideal of a minimal curve, which is a contradiction.

Because I is the ideal of a minimal curve, $a_6 > a_4 + a_5$. Thus

$$a_6 > \max\{a_4 + a_5, a_2 + a_3\}.$$

Now, since $a_1 + a_2 + a_3$ gives the maximal weight of a facet of J, we have $a_1 + a_2 + a_3 \geq a_2 + a_4 + a_6$. Consequently,

$$a_1 + a_2 + a_3 \geq a_2 + a_4 + a_6 > a_2 + a_4 + (a_2 + a_3),$$

using the assumption that $a_2 + a_3 > a_6$. Therefore $a_1 > a_2 + a_4$. Similarly,

$$a_1 + a_2 + a_3 \geq a_3 + a_5 + a_6 > a_3 + a_5 + (a_2 + a_3),$$

so $a_1 > a_3 + a_5$. Hence

$$a_1 > \max\{a_3 + a_5, a_2 + a_4\}.$$

Combining this with the inequalities for a_6 and using Corollary 4.3, we conclude that J is the ideal of a minimal curve, a contradiction.

Thus if J reduces in one step to the ideal of a minimal curve by reducing a facet of maximal weight, the maximal degree of a minimal generator of J is equal to the maximal weight of a facet of J. By Corollary 5.4, this is also the regularity of J. Now the result for all ideals of nonminimal, non-arithmetically Cohen-Macaulay curves follows from the same induction process as in the arithmetically Cohen-Macaulay case. □

With Theorem 5.6, we can read the regularity of a tetrahedral curve straight from the a_i that describe it. With a bit more work, we can also find the graded Betti numbers of the ideal of a tetrahedral curve without any substantial computation. We begin with a lemma that describes how the maximal weight of a facet changes when we do a basic double link. Its proof is similar to several of our earlier arguments in this section.

Lemma 5.7. *Let J be the ideal of a nonminimal tetrahedral curve that is not of the form $(0, r, r, r, r, 0)$, even with the variables permuted. Suppose J is a basic double link of I, and I is obtained by reducing a facet of maximal weight. Then the maximal weight of a facet of J is strictly larger than the maximal weight of a facet of I.*

Proof. Let J be the ideal of the tetrahedral curve $(a_1, a_2, a_3, a_4, a_5, a_6)$, and assume without loss of generality that $a_1 + a_2 + a_3$ gives the maximal weight of a facet. Then I is the ideal of a curve $(a'_1, a'_2, a'_3, a_4, a_5, a_6)$, where $a'_i = \max\{0, a_i - 1\}$. Clearly $a_1 + a_2 + a_3$ is at least as large as the weight of any facet of I, but suppose it is equal to the weight of a facet of I. Without loss of generality, assume that $a_1 + a_2 + a_3 = a'_1 + a_4 + a_5$. Since $a_1 + a_2 + a_3 \geq a_1 + a_4 + a_5$, we conclude that $a_1 = a'_1 = 0$, and $a_2 + a_3 = a_4 + a_5$. Because $a'_1 + a_4 + a_5 = a_4 + a_5$ gives the maximal weight of a facet of I, we have

$$a_4 + a_5 \geq a'_2 + a_4 + a_6 \quad \text{and} \quad a_4 + a_5 \geq a'_3 + a_5 + a_6,$$

so $a_5 \geq a'_2 + a_6$ and $a_4 \geq a'_3 + a_6$. Adding these inequalities and using the fact that $a_2 + a_3 = a_4 + a_5$, we have

$$a_2 + a_3 = a_4 + a_5 \geq a'_2 + a'_3 + 2a_6,$$

and thus $a_6 \leq 1$.

Suppose first that $a_6 = 1$. Because $a_2 + a_3$ is the maximal weight of a facet of J, we have

$$a_2 + a_3 \geq a_2 + a_4 + 1 \quad \text{and} \quad a_2 + a_3 \geq a_3 + a_5 + 1,$$

which implies that $a_3 \geq a_4 + 1$ and $a_2 \geq a_5 + 1$. This contradicts the fact that $a_2 + a_3 = a_4 + a_5$.

Therefore $a_6 = 0$. Then

$$a_4 + a_5 = a_1 + a_4 + a_5 = a_1 + a_2 + a_3 \geq a_2 + a_4 + a_6 = a_2 + a_4;$$

hence $a_5 \geq a_2$. Similarly, $a_4 \geq a_3$. Because $a_2 + a_3 = a_4 + a_5$, $a_5 = a_2$ and $a_4 = a_3$, and J has the form $(0, a_2, a_3, a_3, a_2, 0)$. If $a_2 = a_3$, this contradicts the assumption that J is not of the form $(0, r, r, r, r, 0)$. Otherwise, by Corollary 4.3, J is the ideal of a minimal curve, again a contradiction. □

As a corollary, we obtain a characterization of when the mapping cone resolution of J coming from a basic double link is minimal.

Corollary 5.8. *Let J be the ideal of a nonminimal tetrahedral curve. Assume that J is a basic double link of I, so $J = L \cdot I + (F)$, where L is a linear form, and assume that I is obtained by reducing a facet of J of maximal weight. Set* $\deg F = e$. *Then the mapping cone resolution of J coming from the short exact sequence*

$$0 \to R(-e-1) \to I(-1) \oplus R(-e) \to J \to 0$$

is minimal if and only if J is not the ideal of a curve $(0, r, r, r, r, 0)$ (possibly with the variables permuted).

Proof. First, suppose J is of the form $(0, r, r, r, r, 0)$. Then by Theorem 4.7, F is a minimal generator of I. Therefore the mapping cone resolution cannot be minimal, for $L \cdot F$ will not be one of the minimal generators of J.

Now suppose that J is not the ideal of a curve $(0, r, r, r, r, 0)$. By Lemma 5.7, the maximal weight of a facet of J is strictly greater than the maximal weight of a facet of I. If I is not the ideal of a minimal, non-arithmetically Cohen-Macaulay curve, the maximal weight of a facet of I is equal to the maximal degree in which I has a minimal generator by Lemma 5.6. Therefore the degree of F is at least as large as the degree of the highest degree of a minimal generator of $L \cdot I$. Because all the minimal generators of $L \cdot I$ are divisible by L, and F is not, F is not a redundant generator of J, and its degree is too high to make any of the minimal generators of $L \cdot I$ redundant. Hence the mapping cone resolution is minimal.

If I is not arithmetically Cohen-Macaulay, and I is minimal, then F is not a minimal generator of I by Theorem 4.7. Therefore the degree of F is at least as large as the degree of the minimal generators of $L \cdot I$, and again, F and all of the minimal generators of $L \cdot I$ are minimal generators of J. This proves that the mapping cone resolution is minimal. □

Remark 5.9. We can use Corollary 5.8 to help describe an inductive procedure with which we can easily compute the graded Betti numbers of the ideal of any tetrahedral curve. Suppose J is the ideal of a tetrahedral curve. Using the algorithm from [14], reducing by a facet of maximal weight, we get a sequence of reductions

$$J = J_s \mapsto J_{s-1} \mapsto \cdots \mapsto J_1 \mapsto J_0 = M.$$

If J is not arithmetically Cohen-Macaulay, let M be the ideal of the minimal curve to which the curve corresponding to J reduces. If J is arithmetically Cohen-Macaulay and componentwise linear, let M be the ideal of the trivial curve; that is, $M = (1)$. Finally, if J is arithmetically Cohen-Macaulay and not componentwise linear, suppose J reduces to the ideal of a curve of the form $(0, r, r, r, r, 0)$ but not $(0, r+1, r+1, r+1, r+1, 0)$, and let M be the ideal of $(0, r, r, r, r, 0)$. In all three cases, we know the minimal graded free resolution of M (in the nontrivial cases, from Theorem 4.4 or Theorem 4.7(b)). By Corollary 5.8, the mapping cone resolution of J_r obtained from the short exact sequence induced by the basic double link $J_r = L_r \cdot J_{r-1} + (F_r)$ is minimal. Therefore to get the minimal resolution of J_r, one shifts the minimal resolution of J_{r-1} by one degree and adds a generator of degree $\deg F_r$ and a syzygy of degree $(\deg F_r + 1)$. If $J \neq M$, we can read the maximal degree of a minimal generator (and the corresponding highest degree first syzygy) directly from the maximal weight of a facet of J and then continue the process inductively with the rest of the J_r.

Once we know the reduction sequence, the minimal free resolution of J can be written immediately only from knowledge of the sequence and of M. We illustrate this process in three examples.

Example 5.10. Suppose J is the ideal of the curve $(1, 2, 1, 2, 0, 2)$. We illustrate the reduction procedure and degrees of generators and syzygies at each step.

Curve	**Maximal weight**	**Degree of generator**	**Degree of syzygy**
$(1, 2, 1, 2, 0, 2)$	6	6	7
$(1, 1, 1, 1, 0, 1)$	3	3+1=4	4+1=5
$(0, 0, 0, 1, 0, 1)$	2	2+2=4	3+2=5
$(0, 0, 0, 0, 0, 0)$			

We add in the resolution of R itself, shifted in degree by three because of the three reductions. Therefore the minimal graded free resolution of J is

$$0 \to R(-7) \oplus R(-5)2 \to R(-6) \oplus R(-4)2 \oplus R(-3) \to J \to 0.$$

Example 5.11. Let J be the ideal of the curve $(1, 3, 4, 2, 3, 0)$. Then J is arithmetically Cohen-Macaulay and not componentwise linear; thus it reduces to a curve of the form $(0, r, r, r, r, 0)$, and we know the resolutions of those curves by Theorem 4.7(b).

Curve	**Maximal weight**	**Degree of generator**	**Degree of syzygy**
$(1, 3, 4, 2, 3, 0)$	8	8	9
$(0, 2, 3, 2, 3, 0)$	6	6+1=7	7+1=8
$(0, 2, 2, 2, 2, 0)$			

We add to this the resolution of $(0, 2, 2, 2, 2, 0)$, shifted by two since there were two reductions. Using Theorem 4.7(b), this gives three generators of degree $4 + 2 = 6$ and two syzygies of degree $6 + 2 = 8$. Hence the minimal resolution of J is

$$0 \to R(-9) \oplus R(-8)3 \to R(-8) \oplus R(-7) \oplus R(-6)3 \to J \to 0.$$

Example 5.12. We consider a curve that is not arithmetically Cohen-Macaulay. Let J be the ideal of the curve $(7, 5, 5, 2, 1, 6)$.

Curve	Maximal weight	Degree of generator	Degree of first syzygy
$(7,5,5,2,1,6)$	17	17	18
$(6,4,4,2,1,6)$	14	14+1=15	15+1=16
$(5,3,3,2,1,6)$	11	11+2=13	12+2=14
$(4,2,2,2,1,6)$	10	10+3=13	11+3=14
$(4,1,2,1,1,5)$			

To the generators and syzygies from the reduction, we add the resolution of the minimal curve $(4,1,2,1,1,5)$, shifted by four because of the four reductions. By Theorem 4.4, the resolution of the ideal I of $(4,1,2,1,1,5)$ is

$$0 \to R(-11)^{14} \to R(-10)^{37} \to R(-9)^{24} \to I \to 0.$$

Therefore the resolution of J is

$$0 \to R(-15)^{14} \to R(-18) \oplus R(-16) \oplus R(-14)^{39} \to R(-17) \oplus R(-15) \oplus R(-13)^{26} \to J \to 0.$$

In case of non-arithmetically Cohen-Macaulay curves, part of the preceding discussion can be summarized as follows.

Corollary 5.13. *Let J be the ideal of a tetrahedral curve that is not arithmetically Cohen-Macaulay. Then its minimal free resolution is of the form*

$$0 \to R^{\beta_3}(-e_0-s-2) \to \begin{array}{c} G(-1) \\ \oplus \\ R^{\beta_2}(-e_0-s-1) \end{array} \to \begin{array}{c} G \\ \oplus \\ R^{\beta_1}(-e_0-s) \end{array} \to J \to 0$$

where $G = \bigoplus_{i=1}^{s} R(-e_i)$, $s \geq 0$, and $e_s > \ldots > e_1 > e_0$.

Here, s is the number of steps needed to reduce J to the minimal curve J_0 and $\beta_1, \beta_2, \beta_3 > 0$ are the Betti numbers of J_0 (cf. Theorem 4.4).

Proof. The algorithm from [14] that reduces the curves by using a facet of maximal weight provides a sequence of reductions

$$J = J_s \mapsto J_{s-1} \mapsto \cdots \mapsto J_1 \mapsto J_0$$

where J_0 is a minimal curve. Let e_i be the maximal weight of a facet of the curve J_i if $i > 0$ and let e_0 be the degree of the minimal generators of J_0. Then Lemma 5.7 gives $e_s > \ldots > e_1 > e_0$ and the resolution of J is obtained by using Corollary 5.8 successively. □

A similar description can be given for the arithmetically Cohen-Macaulay tetrahedral curves where we have to distinguish whether the curve is componentwise linear or not. We leave the details to the reader.

6. Tetrahedral Curves with Linear Resolutions

Since the property of having a linear resolution is stronger than that of being componentwise linear, we now turn to the question of which tetrahedral curves have linear resolution. We begin with arithmetically Cohen-Macaulay tetrahedral curves.

Proposition 6.1. *The following are the only arithmetically Cohen-Macaulay tetrahedral curves with linear resolution (up to permutation of the variables):*

(a) $(r, 0, 0, 0, 0, 0)$ *for some* $r \geq 1$*;*
(b) $(1, 1, 0, 1, 0, 0)$ *(this is the union of three non-coplanar lines in* $\mathbb{P}^3$ *meeting at one point);*
(c) $(1, 1, 1, 1, 1, 1)$*;*
(d) $(2, 1, 0, 1, 0, 1)$*;*
(e) $(2, 1, 1, 1, 1, 2)$

Proof. It is easy to check that these curves do have linear resolution. We have to check that they are the only arithmetically Cohen-Macaulay tetrahedral curves with this property (up to permutation of the variables).

Let C be an arithmetically Cohen-Macaulay tetrahedral curve. We reduce using facets of maximal weight until one of the following happens: either (i) we obtain a curve of type $(0, r, r, r, r, 0)$ (up to permutation of the variables), or (ii) we obtain a *plane* curve of degree 1, 2 or 3 (which is then one step away from the trivial curve via facets of maximal weight). In either of these cases, each step in the reduction, passing from some J to a smaller curve with ideal I, used a form F that was not a minimal generator of I, thanks to Theorem 4.7. It then follows from Corollary 3.3 and Theorem 4.7 (b) that in case (i), I_C does not have a linear resolution.

So without loss of generality, we may reduce to a plane curve of degree 1, 2 or 3 via facets of maximal weight, each time using a form F that is not a minimal generator of the smaller curve. If we arrive at a plane curve of degree 2 or 3, then again by Corollary 3.3, I_C does not have a linear resolution since a complete intersection of type $(1, 2)$ or type $(1, 3)$ does not have a linear resolution. So I_C reduces to a line via facets of maximal weight.

So we may work backwards, beginning with the curve $I = (1, 0, 0, 0, 0, 0)$. In order to form $J = L \cdot I + (F)$ and have the result have a linear resolution, we need that $\deg F = 2$. Using (A) we obtain $(2, 0, 0, 0, 0, 0)$. Using (C) we obtain $(1, 1, 0, 1, 0, 0)$ (or $(1, 0, 0, 1, 0, 1)$ or $(1, 1, 0, 0, 0, 1)$, which are equivalent). (B) can only repeat the result of (A), and (D) repeats the result of (C) (up to permutation).

For the next step we have to pass from $(2, 0, 0, 0, 0, 0)$ or $(1, 1, 0, 1, 0, 0)$ to the next curve using F of degree 3. If we start with $(2, 0, 0, 0, 0, 0)$ and use (A) or (B), clearly a_1 becomes 3 so the remaining entries must stay 0, and we can only obtain $(3, 0, 0, 0, 0, 0)$. If we start with $(2, 0, 0, 0, 0, 0)$ and use (C) or (D) we obtain $(2, 1, 0, 1, 0, 1)$ or $(2, 0, 1, 0, 1, 1)$, which are equivalent. If we start with $(1, 1, 0, 1, 0, 0)$ then the only permissible basic double link that uses a form of degree 3 uses (D), and we obtain $(1, 1, 1, 1, 1, 1)$.

Passing to the next step, we need to use a form F of degree 4. Starting with $(3, 0, 0, 0, 0, 0)$, the only possibility is to use (A) or (B) and pass to $(4, 0, 0, 0, 0, 0, 0)$. If instead we start with $(2, 1, 0, 1, 0, 1)$, the only possibility is to use (D), from which we obtain $(2, 1, 1, 1, 1, 2)$. If we start with $(1, 1, 1, 1, 1, 1)$, none of the operations produces a result with linear resolution since all forms F will have degree 6.

For the next step, we need to use a form F of degree 5. From $(4, 0, 0, 0, 0, 0)$ it is clear that we can pass only to $(5, 0, 0, 0, 0, 0)$. From $(2, 1, 1, 1, 1, 2)$, none of the operations produces a result with linear resolution.

It is clear that from $(r,0,0,0,0,0)$ we can obtain $(r+1,0,0,0,0,0)$. Also, we know that if I fails to have a linear resolution then so does J, so once we lose this property we can never get it back. Hence this completes the proof. $\square$

We now turn to non arithmetically Cohen-Macaulay tetrahedral curves. Since basic double linkage preserves the even liaison class ([12], [13]), it is convenient to look within a fixed even liaison class. Our first observation is that it can happen that there are fewer non-minimal tetrahedral curves in the class than one might expect.

Proposition 6.2. *Let $I = (a_1, a_2, a_3, a_4, a_5, a_6)$ be a minimal tetrahedral curve. Assume that*

$$\begin{array}{rcl} a_1 & > & \max\{a_3+a_5+2, a_2+a_4+2\} \text{ and} \\ a_6 & > & \max\{a_4+a_5+2, a_2+a_3+2\} \end{array}$$

Then the even liaison class of I contains no non-minimal tetrahedral curves that reduce to I.

Proof. It follows from Corollary 4.3. If the even liaison class contained a non-minimal tetrahedral curve that can be reduced via (A), (B), (C) and (D) of Proposition 4.1 to I, then in the last step we pass from a tetrahedral curve J to I, where the 6-tuple corresponding to J has three of its entries equal to the corresponding ones of I, and up to three others (and exactly three others, if the entries are non-zero) that are one more than the corresponding ones of I. Without loss of generality, suppose that $J = (a_1+1, a_2+1, a_3+1, a_4, a_5, a_6)$. But the stated hypothesis then gives, via Corollary 4.3, that J is minimal. Hence J cannot have arisen from I by basic double linkage. $\square$

Remark 6.3. Since we do not yet have a good understanding of the Hartshorne-Rao module of a tetrahedral curve, we do not know if there may be another 6-tuple that is in the same even liaison class, also minimal, but which does allow ascending tetrahedral curves. Still, there are some cases where we know that this does not happen. For example, it was noted in [14], Remark 5.5, that the curve $(m,0,0,0,0,k)$, with $m,k \geq 2$, is the unique minimal curve in its even liaison class, thanks to the main result of [12]. Since this curve satisfies the hypothesis of Proposition 6.2, it is in fact the *only* tetrahedral curve in its even liaison class.

We also remark that if a minimal tetrahedral curve admits one basic double link of the type (A), (B), (C) or (D), then it allows infinitely many (sequentially), and there are infinitely many tetrahedral curves in the class.

We have seen in Proposition 6.1 that there are infinitely many 6-tuples representing arithmetically Cohen-Macaulay curves with linear resolution, but if we identify those that are a multiple of a single line then there are only finitely many. We now show that the latter is true also for non arithmetically Cohen-Macaulay curves. We begin with the even liaison class that we believe has the largest number of tetrahedral curves with linear resolution.

Proposition 6.4. *Let $\mathcal{L}$ be the even liaison class of two skew lines. Among tetrahedral curves, this means $(1,0,0,0,0,1)$, $(0,1,0,0,1,0)$ or $(0,0,1,1,0,0)$. Then up to permutation of the variables, the following are the only tetrahedral curves in $\mathcal{L}$ with linear resolution:*

(a) $(1, 0, 0, 0, 0, 1)$
(b) $(2, 1, 0, 0, 0, 1)$
(c) $(3, 1, 0, 1, 0, 1)$
(d) $(2, 2, 0, 0, 0, 2)$
(e) $(2, 1, 1, 1, 0, 1)$
(f) $(3, 2, 0, 1, 1, 2)$
(g) $(3, 2, 1, 1, 2, 3)$

Proof. Beginning with the ideal, I, of two skew lines, we must perform a basic double link following the guidelines of Proposition 4.1, but using a form F of degree 3 (since the generators of I have degree 2). So, for instance, from $(1, 0, 0, 0, 0, 1)$ the only options are to use $F = b^2c$ or b^2d for type (A), a^2c or a^2d for type (B), $ad2$ or $bd2$ for type (C), and $ac2$ or $bc2$ for type (D), and we obtain the permutations of (b) having either first or last entry equal to 2. Continuing in this way (taking the next basic double link using F of degree 4), one can exhaust all the possibilities. We leave the details to the reader. □

Theorem 6.5. *Let $\mathcal{L}$ be the even liaison class of a non arithmetically Cohen-Macaulay tetrahedral curve. Then $\mathcal{L}$ has only finitely many tetrahedral curves with linear resolution.*

Proof. Let J be the ideal of a tetrahedral curve in $\mathcal{L}$ that has a linear resolution. We have seen that J can be reduced to a minimal tetrahedral curve by a sequence of reductions of the form (A), (B), (C) or (D) as given in Proposition 4.1, and that we can do this always using a facet of maximal weight (see also Definition 4.2). Let $C_0 = (a_1, a_2, a_3, a_4, a_5, a_6)$ be the minimal curve so obtained. We have seen in Theorem 4.4 that I_{C_0} has a linear resolution, and that the degree of its minimal generators is $a_1 + a_6$.

Our strategy will be to show that in any sequence of basic double links that preserves the linearity of the resolutions, we cannot use any of (A), (B), (C) or (D) more than once.

We first claim that in any such sequence of basic double links, all the intermediate tetrahedral curves $C_1, C_2, \dots$ between the ideals I_{C_0} and J have linear resolution. Indeed, suppose that J reduces to I, and that I fails to have a linear resolution. Assume that the minimal generators of J all have degree d. Since C_0 is not arithmetically Cohen-Macaulay, we have seen (Theorem 4.7) that the polynomial F used in the reduction is not a minimal generator of I, but by construction it is a minimal generator of J (it is the only generator that does not have as a factor the linear form used in the basic double link); hence it has degree d. But then from the exact sequence

$$0 \to R(-d-1) \to I(-1) \oplus R(-d) \to J \to 0$$

it is clear that no splitting can occur in the mapping cone to restore linearity to the non-linear resolution of I, no matter where the non-linearity occurs in the resolution.

Consequently, if we work backwards, starting with I_{C_0} and building up to J with basic double links, the first basic double link must use a polynomial F of degree $a_1 + a_6 + 1$, the next a polynomial of degree $a_1 + a_6 + 2$, and so on.

If all entries $a_i > 0$, $1 \le i \le 6$, then the result is not hard to see. Indeed, suppose without loss of generality that the first basic double link is of type (A) in Proposition 4.1. Then $C_1 = (a_1 + 1, a_2 + 1, a_3 + 1, a_4, a_5, a_6)$ and we have that the first three entries give the facet of maximal weight. Hence $a_1 + a_2 + a_3 + 3 = a_1 + a_6 + 1$. It is clear that we cannot use type (A) again, since then the new curve C_2 will have the sum of the first three entries be strictly greater than $a_1 + a_6 + 2$, while we would need equality. But in fact, any other type that we use increases one of a_1, a_2, a_3 by 1, so that we can *never* return to use type (A). But the same happens with the type used in the next step – it can be used at most once. Continuing in this way, we see that at most four basic double links can be used in order to preserve the linearity of the resolution, so the result follows.

The only chance for the result to fail, then, is if some entries a_i are 0, and remain 0 even after the basic double link (a possibility allowed in Proposition 4.1). Suppose without loss of generality that $a_6 = \max\{a_i\}$. We know from Proposition 4.1 and Corollary 4.3 that also $a_1 > 0$, and that

$$\begin{array}{rcl} a_1 & > & \max\{a_3 + a_5, a_2 + a_4\} \text{ and} \\ a_6 & > & \max\{a_4 + a_5, a_2 + a_3\}. \end{array}$$

Suppose without loss of generality, again, that I_{C_1} is obtained via the basic double link described in (A) of Proposition 4.1. A priori, I_{C_1} could be any of the following tetrahedral curves:

(i) $(a_1 + 1, 0, 0, a_4, a_5, a_6)$ (here $a_2 = a_3 = 0$);
(ii) $(a_1 + 1, a_2 + 1, 0, a_4, a_5, a_6)$ (here $a_2 \geq 0, a_3 = 0$);
(iii) $(a_1 + 1, 0, a_3 + 1, a_4, a_5, a_6)$ (here $a_2 = 0, a_3 \geq 0$);
(iv) $(a_1 + 1, a_2 + 1, a_3 + 1, a_4, a_5, a_6)$;

Notice, though, that in case (i) the new curve C_1 is again a minimal curve (Corollary 4.3), so it is not in the same even liaison class. Hence (i) does not happen. Also, in case (iv) it is easy to see as above that we can never use (A) again, since the sum of the first three entries is too big to use (A) to get C_2, and any of types (A)-(D) increases at least one of a_1, a_2 or a_3 by 1 so $\deg F$ can never "catch up" to get subsequent C_i. By symmetry, there is no difference between cases (ii) and (iii), so without loss of generality let us assume that (ii) holds.

Since $\deg F = a_1 + a_6 + 1 = (a_1 + 1) + (a_2 + 1)$ in the first basic double link, we observe that $a_6 = a_2 + 1$, so

$$I_{C_1} = (a_1 + 1, a_2 + 1, 0, a_4, a_5, a_2 + 1).$$

Now, we have seen that the next basic double link (to pass from C_1 to C_2) uses a polynomial F of degree $a_1 + a_6 + 2$ in order to preserve linearity. If we were to use another basic double link of type (A), though, we would have to increase the first entry and the second entry by 1. Hence we would have

$$\deg F = (a_1 + 2) + (a_2 + 2) + (0 \text{ or } 1) = a_1 + a_6 + 2 = a_1 + a_2 + 3,$$

which is impossible. So we cannot use (A) again, at least not now.

We will suppose that we use (B) for the second basic double link, and carefully analyze the possibilities. The other options for the second basic double link are analyzed in a similar way. If we use (B) for the second basic double link, then we must increase either the fourth entry or the fifth entry (or both) by 1, since otherwise we have $a_4 + a_5 = 0 \geq a_6$, which is impossible. Hence at least two entries (including a_1) increase by 1, and as before, if all three entries increase by 1 then we can never use (B) again. So to preserve hope of using (B) again, we have two cases: (i) $a_4 = 0$ and remains 0 after the first application of (B), and (ii) $a_5 = 0$ and remains 0 after the first application of (B).

In case (i), we have

$$\begin{array}{rcl} I_{C_1} & = & (a_1+1, a_2+1, 0, 0, a_5, a_2+1) \\ I_{C_2} & = & (a_1+2, a_2+1, 0, 0, a_5+1, a_2+1) \end{array}$$

But then we have

$$(a_1+2) + 0 + (a_5+1) = a_1 + a_6 + 2,$$

so $a_6 = a_5 + 1$ and hence $a_2 = a_5$. Thus case (i) gives us

$$\begin{array}{rcl} I_{C_0} & = & (a_1, a_2, 0, 0, a_2, a_2+1) \\ I_{C_1} & = & (a_1+1, a_2+1, 0, 0, a_2, a_2+1) \\ I_{C_2} & = & (a_1+2, a_2+1, 0, 0, a_2+1, a_2+1) \end{array}$$

As before, we cannot use (B) again unless we use (C) at some point and preserve $a_4 = 0$, since (A) and (D) both increase either a_1 or a_5 by 1. And we cannot use (A) unless we use (D) first.

In case (ii) we have

$$\begin{array}{rcl} I_{C_1} & = & (a_1+1, a_2+1, 0, a_4, 0, a_2+1) \\ I_{C_2} & = & (a_1+2, a_2+1, 0, a_4+1, 0, a_2+1) \end{array}$$

But then we have

$$(a_1+2) + (a_4+1) + 0 = a_1 + a_6 + 2,$$

so $a_6 = a_4 + 1$ and hence $a_2 = a_4$. Thus case (ii) gives

$$\begin{array}{rcl} I_{C_0} & = & (a_1, a_2, 0, a_2, 0, a_2+1) \\ I_{C_1} & = & (a_1+1, a_2+1, 0, a_2, 0, a_2+1) \\ I_{C_2} & = & (a_1+2, a_2+1, 0, a_2+1, 0, a_2+1) \end{array}$$

As before, we cannot use (B) again unless we use (D) at some point and preserve $a_5 = 0$, since (A) and (C) both increase either a_1 or a_4 by 1. And we cannot use (A) unless we use (D) first.

Now we consider the third basic double link (i.e. passing from C_2 to C_3). In case (i) above, we have two options: (i-a) to use (C) next, and (i-b) to use (D) next. In case (i-a), we obtain

$$a_1 + a_6 + 3 = (a_2+2) + (0 \text{ or } 1) + (a_2+2),$$

which gives $a_1 = a_2 + (0 \text{ or } 1)$, and since $a_1 > a_2$ we have $a_1 = a_2 + 1$. But this means that our use of (C) increased a_4 from 0 to 1, and we can never use (B) again. Similarly, since at this point the second, fourth and sixth entries are > 0, we can never use (C) again either. And at this stage we cannot use (A) again unless we use (D) and preserve $a_3 = 0$. Hence case (i-a) gives

$$\begin{array}{rcl} I_{C_0} & = & (a_2+1, a_2, 0, 0, a_2, a_2+1) \\ I_{C_1} & = & (a_2+2, a_2+1, 0, 0, a_2, a_2+1) \\ I_{C_2} & = & (a_2+3, a_2+1, 0, 0, a_2+1, a_2+1) \\ I_{C_3} & = & (a_2+3, a_2+2, 0, 1, a_2+1, a_2+2) \end{array}$$

and the only possible fourth basic double link is (D).

In case (i-b) we have

$$a_1 + a_6 + 3 = (0 \text{ or } 1) + (a_2+2) + (a_2+2)$$

so since $a_1 > a_2$, we have that (D) increases a_3 from 0 to 1, and $a_1 = a_2 + 1$. In this case we can never use (A) or (D) again. Hence case (i-b) gives

$$\begin{aligned} I_{C_0} &= (a_2+1, a_2, 0, 0, a_2, a_2+1) \\ I_{C_1} &= (a_2+2, a_2+1, 0, 0, a_2, a_2+1) \\ I_{C_2} &= (a_2+3, a_2+1, 0, 0, a_2+1, a_2+1) \\ I_{C_3} &= (a_2+3, a_2+1, 1, 0, a_2+2, a_2+2) \end{aligned}$$

Furthermore, since the fourth basic double link uses $\deg F = a_1 + a_6 + 4$, it is not hard to see that at this stage we cannot use (B) again. Hence the only possible fourth basic double link uses (C).

In both cases (i-a) and (i-b), it is not hard to see that the fourth basic double link forces the last remaining 0 entry to become 1 (since we need $\deg F = a_1 + a_6 + 4 = 2a_2 + 6$), and hence we cannot use any of the four types of basic double links and preserve the linearity of the resolution. Case (ii), and the other cases, are proven similarly. □

7. The Generic Initial Ideal of a Tetrahedral Curve

In this section, we will assume that the characteristic of k is zero. With this hypothesis, generic initial ideals are always strongly stable. We will take generic initial ideals with respect to the reverse-lexicographic order; using this order allows us to use some nice relationships from [2] between an ideal and its gin.

Using Proposition 5.2, it is easy to describe the minimal generating set of $\operatorname{gin}(J)$ when J is the ideal of an arithmetically Cohen-Macaulay tetrahedral curve.

Proposition 7.1. *Let J be the ideal of an arithmetically Cohen-Macaulay tetrahedral curve with lowest degree minimal generator in degree d_0.*

(a) *If J is componentwise linear and has minimal generators in degrees $d_0 \leq \cdots \leq d_s$, then*
$$\operatorname{gin}(J) = (a^{d_0}, a^{d_0-1}b^{d_1-d_0+1}, \ldots, a^{d_0-p}b^{d_p-d_0+p}, \ldots, b^{d_s}).$$
In particular, $s = d_0$, and J has $d_0 + 1$ minimal generators.

(b) *Suppose J is not componentwise linear and that J has g minimal generators in lowest degree d_0 and h minimal generators in degree $d_0 + 1$. Then $\operatorname{gin}(J)$ has minimal generating set with the monomials in $S = \{a^{d_0}, a^{d_0-1}b, \ldots, a^{d_0-(g-1)}b^{g-1}\}$, the first $h + g - 1$ monomials of degree $d_0 + 1$ not divisible by any element of S, and then elements of higher degree. For each minimal generator of J of degree higher than $d_0 + 1$, there is a minimal generator a^ib^j of $\operatorname{gin}(J)$ of the same degree with the powers on a decreasing down to zero.*

Proof. Since J has codimension two, and R/J is Cohen-Macaulay, J and $\operatorname{gin}(J)$ have projective dimension two. Because $\operatorname{gin}(J)$ is strongly stable, it is generated by monomials of the form a^ib^j, including a pure power of both a and b; note that by the Eliahou-Kervaire resolution, any minimal generator of $\operatorname{gin}(J)$ involving c or d would contradict the projective dimension being two. Moreover, any stable ideal in two variables is a lexicographic ideal. It follows immediately that
$$\operatorname{gin}(J) = (a^{d_0}, a^{d_0-1}b^{q_1}, a^{d_0-2}b^{q_2}, \ldots, b^{q_t}).$$

Suppose first that J is componentwise linear. Then J and $\text{gin}(J)$ have the same graded Betti numbers and therefore minimal generators of the same degree. The lowest degree generator of $\text{gin}(J)$ is a^{d_0}. The second lowest has degree d_1, and thus it is $a^{d_0-1}b^{d_1-(d_0-1)}$, and so on for the others. All the generators have the form $a^{d_0-r}b^{d_r-(d_0-r)}$, where $0 \leq r \leq s$. Note that the exponent on a decreases by one as r increases by one. Since the minimal generator of $\text{gin}(J)$ of highest degree is a pure power of b, $s = d_0$, and $\text{gin}(J)$ has d_0+1 minimal generators. Because J and $\text{gin}(J)$ have the same graded Betti numbers, J also has d_0+1 minimal generators.

Part (b) follows from Proposition 5.2. The $(g-1)$ additional generators in degree d_0+1 come from the fact that (J_{d_0}) does not have a linear resolution, requiring us to add $g-1$ generators and syzygies of degree d_0+1 when we move to the gin. $\square$

Example 7.2. Let I be the ideal of the curve $(1, 2, 2, 2, 1, 2)$. Then I has minimal resolution

$$0 \to R(-7) \oplus R(-6)2 \oplus R(-5) \to R(-6) \oplus R(-5)2 \oplus R(-4)2 \to I \to 0.$$

Note that I is componentwise linear. Consequently,

$$\text{gin}(I) = (a4, a^3b, a^2b3, ab4, b6).$$

Suppose now that J is the ideal of the curve $(2, 1, 4, 1, 1, 3)$. J is not componentwise linear, and it has minimal resolution

$$0 \to R(-9) \oplus R(-8) \oplus R(-7)2 \to R(-8) \oplus R(-7) \oplus R(-6) \oplus R(-5)2 \to J \to 0.$$

Thus by Proposition 7.1, $\text{gin}(J)$ must have two minimal generators of degree five and two minimal generators of degree six plus generators of higher degree. Therefore

$$\text{gin}(J) = (a5, a^4b, a^3b3, a^2b4, ab6, b8).$$

We now turn to the generic initial ideal of a non arithmetically Cohen-Macaulay tetrahedral curve. We begin with a lemma that says that it is enough to determine the generic initial ideal of the minimal curve in the even liaison class.

Lemma 7.3. *Let J, I be the ideals of non-arithmetically Cohen-Macaulay tetrahedral curves. Assume $J = LI + (F)$ is a basic double link of I where $e := \deg F$ is the maximal weight of a facet of J. Then we have for the generic inital ideals*

$$\text{gin}(J) = a\,\text{gin}(I) + b^e.$$

Proof. By abuse of notation let us denote by I and J the ideals obtained from I and J after a general change of coordinates. Then we have that $a\,\text{in}(I) \subset \text{in}(J)$, hence $a\,\text{gin}(I) \subset \text{gin}(J)$. Since $\text{gin}(J)$ is stable of codimension two, it must contain a power of b. We know by [2] that the Castelnuovo-Mumford regularity of J is e. Therefore, we get

$$a\,\text{gin}(I) + (b^e) \subset \text{gin}(J).$$

But $a\,\text{gin}(I) + (b^e)$ is a basic double link of $\text{gin}(I)$. Since I and J are componentwise linear, their graded Betti numbers agree with the ones of their generic initial ideals. It follows that $a\,\text{gin}(I) + (b^e)$ and $\text{gin}(J)$ have identical graded Betti numbers, thus these ideals agree. $\square$

While we are not yet able to determine the generic initial ideal of an arbitrary minimal tetrahedral curve, we are able to do it for arithmetically Buchsbaum tetrahedral curves. It was shown in [14] that up to a permutation of variables, a minimal arithmetically Buchsbaum tetrahedral curve is of the form $I_r = (r, 0, r-1, r-1, 0, r)$. It is not hard to use liaison addition (cf. [15], [8]) to show the recursive relation

$$I_{r+1} = (ac) \cdot I_r + (bd)^r \cdot I_1. \tag{7.1}$$

(One shows the inclusion $\supseteq$ and then argues that the two ideals are both saturated and define curves of the same degree.)

Proposition 7.4. *The generic initial ideal of a minimal arithmetically Buchsbaum tetrahedral curve $I_r = (r, 0, r-1, r-1, 0, r)$ is determined recursively by the following:*

(a) $\mathrm{gin}(I_1) = (a2, ab, b2, ac)$.
(b) $\mathrm{gin}(I_{r+1}) = (a2) \cdot \mathrm{gin}(I_r) + (ab^{2r+1}, b^{2r+2}, a^{r+1}b^r c)$.

Proof. Part (a) is immediate, since I_1 has codimension two and is componentwise linear, with four minimal generators all in degree 2, and $\mathrm{gin}(I_1)$ is strongly stable. For part (b), we have from (7.1) that

$$(a2) \cdot \mathrm{gin}(I_r) \subseteq \mathrm{gin}(I_{r+1}).$$

We also know that the number of minimal generators of I_r is $3r+1$, all of degree $2r$, and in fact that I_r has a linear resolution:

$$0 \to R(-2r-2)^r \to R(-2r-1)^{4r} \to R(-2r)^{3r+1} \to I_r \to 0$$

(cf. Theorem 4.4). Hence $\mathrm{gin}(I_r)$ has the same resolution, since I_r is componentwise linear. From the above inclusion, we have $3r+1$ minimal generators for $\mathrm{gin}(I_{r+1})$, and it is clear that also ab^{2r+1} and b^{2r+2} are minimal generators, since $\mathrm{gin}(I_r)$ has codimension two and is strongly stable. We have only to prove that the last minimal generator is $a^{r+1}b^r c$ (and not $a^{2r}c2$, for instance).

Let $C := C_{r+1}$ be the tetrahedral curve with ideal $I_{r+1} = (r+1, 0, r, r, 0, r+1)$. We know that $\deg C = 2(r+1)2$, and that the Hartshorne-Rao module $M(C)$ has dimension $r+1$ and is concentrated in degree $2r$ (cf. [3]).

Let L be a general linear form defining a plane H in $\mathbb{P}^3$ and let $t \in \mathbb{Z}$. From the exact sequence

$$0 \to (I_C)_{t-1} \to (I_C)_t \to (I_{C\cap H})_t \to M(C)_{t-1} \to 0$$

(where the last "$\to 0$" comes because $C = C_{r+1}$ is arithmetically Buchsbaum), we see that

$$\dim(I_{C\cap H})_t = \begin{cases} 0 & \text{if } t \le 2r; \\ r+1 & \text{if } t = 2r+1. \end{cases}$$

It follows that the h-vector of $C \cap H$ begins

$$(1, 2, 3, \dots, 2r, 2r+1, r+1, \dots).$$

But these entries already add up to $\deg C = 2(r+1)2$, so this is the entire h-vector. It follows that in the quotient ring $S = R/(L) \cong k[a, b, c]$, we have

$$\mathrm{gin}(I_{C\cap H}) = (a^{2r}, a^{2r-1}b, \dots, a^{r+1}b^r, \dots)$$

where the entries up to $a^{r+1}b^r$ are all of degree $2r+1$ and the remaining entries (not written) are of degree $2r+2$.

Since I_C is saturated, without loss of generality we may reduce modulo d and work in the ring $S = k[a, b, c]$. Let $I = [I_{r+1} + (d)]/(d)$. We will now apply a result of [5], section 2. They define (with our notation)

$$I^j = \mathrm{im}[(I : c^j) \to S \to S/(c)].$$

We take $j = 1$ and assume that we have a general change of coordinates, and have taken the initial ideal. Lemma 2.6 and Lemma 2.7 of [5] combine to give that

$$a^{i_0}b^{i_1} \in \mathrm{gin}(I1) \iff a^{i_0}b^{i_1}c \in \mathrm{gin}(I).$$

From the information above about C and $C \cap H$, it is clear that $a^{r+1}b^r \in \mathrm{gin}(I1)$; hence it follows that $a^{r+1}b^r c \in \mathrm{gin}(I)$, and we have finished. □

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