CURVES OF DEGREE TWO AND ROPES ON A LINE: THEIR IDEALS AND EVEN LIAISON CLASSES

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Abstract. In this paper we investigate degree two curves of arbitrary codimension. This requires to study also ropes supported on a line. After establishing several characterizations of ropes supported on a line we describe their homogeneous ideals and Hartshorne-Rao modules; in particular we characterize the arithmetically Buchsbaum ropes. Then we describe the even Gorenstein liaison classes of the non-arithmetically Buchsbaum, non-degenerate ropes generalizing a well-know result of Migliore on double lines of codimension two. The result relies on the explicit description of certain arithmetically Gorenstein curves with maximal tangent spaces at each point. As a consequence, we can decide if two curves of degree two belong to the same Gorenstein liaison class. Finally, we show as an application how ropes can be used in order to construct quasi-extremal curves.

Contents

1. Introduction

The curves of smallest degree in projective space are lines where we mean by a curve a closed locally Cohen-Macaulay subscheme of pure dimension one. Let us consider the next case, i.e. curves of degree two. Such a curve is either an irreducible conic, a pair of two lines, or a double line, the latter being the most interesting. Double lines in $\mathbb{P}^3$ are completely described by Ferrand's construction (cf. [10]). Their liaison classes have

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been characterized by Migliore in [17]. The goal of this paper is to extend these results to double lines of arbitrary codimension. In order to achieve this we are forced to consider not only double lines, but more generally ropes supported on a line.

Let Y be a smooth, irreducible curve. An α–rope C on Y is a curve which is a multiplicity α structure on Y satisfying $(I_Y)^2 \subset I_C \subset I_Y$. Ropes have been studied, for example, in [7], [1], [19]. In this paper, we are interested in ropes whose support is a line.

For the rest of this introduction a rope always means a rope supported on a line. Every such $(n-k)$–rope $C \subset \mathbb{P}^n$ is uniquely determined by the following data: the supporting line L and an injective morphism

$$\varphi_{B_C} : \oplus_{j=1}^{k} \mathcal{O}_L(-\beta_j - 1) \to \mathcal{O}_L^{n-1}(-1)$$

which does not drop rank (cf. Theorem 2.4).

We use this description in order to study the ropes from the point of view of liaison theory. While classically only complete intersection were used in order to link two subschemes directly, recent results suggest that a more satisfactory liaison theory is obtained by allowing arithmetically Gorenstein schemes to give the direct links (cf., e.g., [15], [5], [20] and [13]). We adopt this point of view. Thus, liaison always means Gorenstein liaison in this paper.

Observe, that the degree of a non-degenerate arithmetically Gorenstein subscheme of $\mathbb{P}^n$ is at least $n+1$. Thus, the residual scheme of a non-degenerate double line has degree $\geq n-1$. This simple observation shows that in contrast to the situation in $\mathbb{P}^3$, it is not sufficient to consider only double lines for determining the even liaison classes of double lines. But we show that there are arithmetically Gorenstein curves which link ropes to ropes. Combining this with our explicit information on the ropes we can prove:

Theorem 1.1. *Let $C, E \subset \mathbb{P}^n = \operatorname{Proj} R$ be two non-degenerate ropes of the same degree and assume that the arithmetic genus of C is $< \frac{n-1}{2}$. Then the following conditions are equivalent:*

(a) *C and E belong to the same even liaison class;*
(b) *The Hartshorne-Rao modules of C and E are isomorphic as graded R–modules,*
(c)
 1. *C and E have the same line L as support;*
 2. *C and E have the same arithmetic genus;*
 3. *the corresponding maps $\varphi_{B_C}, \varphi_{B_E} : \oplus_{j=1}^{k} \mathcal{O}_L(-\beta_j - 1) \to \mathcal{O}_L^{n-1}(-1)$ differ by an automorphism of the target $\mathcal{O}_L^{n-1}(-1)$.*

For $n = 3$ this statement specializes to the main result in [17].

Actually, we show a more general result (cf. Theorem 5.4). It says that two ropes of the same degree, which are not arithmetically Buchsbaum, belong to the same even liaison class if and only if their Hartshorne-Rao modules are isomorphic. While this result is in codimension two a special case of a general theorem of Rao (cf. [27]), it provides (in higher codimension) some support to the hoped-for generalization of Rao's theorem. It is one of the main open problems of liaison theory to decide if this so-called Rao's correspondence is true for Gorenstein liaison (it is not true for complete intersection liaison). Positive results in this direction have been obtained mainly for arithmetically Cohen-Macaulay curves (cf., e.g., [15], [5], [20], [13]) and for arithmetically Buchsbaum curves (cf., e.g., [16], [13], [6]) by using the theory developed in [15], which allows to view Gorenstein liaison as a theory about divisors on arithmetically Cohen-Macaulay schemes being locally Gorenstein in codimension one, and the classification of arithmetically Buchsbaum divisors on rational normal scrolls (cf. [23]). Note that we cannot use this approach because ropes are typically not generically Gorenstein.

Our cleanest result from a liaison theoretic point of view is Theorem 5.7 which completely describes the even liaison classes of curves of degree two.

Families of ropes will be studied in a subsequent paper [25].

The paper is organized as follows. In Section 1, after comparing different definitions of general ropes we restrict ourselves to ropes supported on a line. In particular, we obtain a very precise description of the homogeneous ideal. Furthermore, we present geometric descriptions of ropes following ideas of C. Segre and J. Harris.

The Hartshorne-Rao module of a rope is computed in Section 3. This result is then used to determine the arithmetically Buchsbaum ropes.

Section 4 is devoted to the construction of a family of non-degenerate arithmetically Gorenstein curves in $\mathbb{P}^n$, supported on a line, having degree $n+1$ and a tangent space of dimension n at each point of the support. In Section 5 we use the curves of this family to link ropes to each other. Here we prove our main results on the even liaison classes.

The final section is devoted to the construction of curves with maximal cohomology in non-negative degrees, subject to the condition that the general hyperplane section is non-degenerate (cf. [8]). Such curves were called quasi-extremal curves in [26] where examples have been constructed algebraically. Here we give a geometric construction of quasi-extremal curves as suitable unions of an $(n-1)$–rope and a plane curve.

Our notation is mainly standard and follows [12]. For definitions, basic results and extensive background on liaison theory we refer to [18].

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2. Characterizations of ropes supported on a line

In this section, we provide some characterizations of an α–rope. Then we specialize to ropes supported on a line where we obtain more precise results. Finally, we describe ropes as a multiple structure on a scroll following a classical construction by C. Segre.

Throughout this paper we will use the following notation. We put $r := n-2$ and denote by R the polynomial ring $K[x_0, \ldots, x_r, t, u]$ over an arbitrary field. It will be convenient to assume that the supporting line $L \subset \mathbb{P}^n := \mathrm{Proj}(R)$ is $L = \mathrm{Proj}(K[t, u])$.

Now, we begin by recalling the definition of an α–rope following [19]. For a more general definition applying also to non-embedded curves, we refer to [7], Definition 1. Let Y be a smooth and irreducible curve in $\mathbb{P}^n$ with saturated homogeneous ideal I_Y (the support of the rope), and let C be a subscheme of $\mathbb{P}^n$ with homogeneous ideal I_C.

Definition 2.1. A curve $C \subset \mathbb{P}^n$ is said to be an *α–rope* supported on Y if its homogeneous ideal I_C satisfies $I_Y^2 \subseteq I_C \subseteq I_Y$ and if C is a locally Cohen–Macaulay multiplicity α structure on Y.

Remark 2.2. It is easy to see that an α–rope $C \subset \mathbb{P}^n$ supported on Y corresponds naturally to a rank $\alpha - 1$ subbundle E of the normal bundle $\mathcal{N}_Y$ of Y via the exact sequence

$$0 \to E^* \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_Y \to 0.$$

This also shows that the definitions of an α–rope in [7] and above agree for embedded ropes.

Next, we show that the definition above is equivalent to a generalization of the definition of *doubling* given by M. Boratynski and S. Greco in [3].

Proposition 2.3. *With the notation above, the 1-dimensional subscheme $C \subset \mathbb{P}^n$ is an α–rope on Y if and only if the following conditions are satisfied:*

(1) $I_Y^2 \subseteq I_C \subseteq I_Y$;
(2) I_C *is unmixed;*
(3) $l(R_{I_Y}/I_C R_{I_Y}) = \alpha$.

Proof. It is enough to prove that $l(R_{I_Y}/I_C R_{I_Y}) \cdot \deg Y$ is the degree of the rope C. But this follows from the "multiplicity formula" (see, e.g., [21], 23.5). □

In [19], the authors describe the ideal I_C of a rope C, proving that every $(n-k)$–rope supported on Y is obtained by starting with an ideal $J = (I_Y^2, F_1, \ldots, F_k)$ and then, possibly, removing embedded points.

In this paper we are interested in ropes supported on a line L. The next result shows that in this case embedded points do not occur. Without loss of generality, we can fix the line $L = V(x_0, \ldots, x_r) = \mathrm{Proj}(S)$ where $S = K[t, u]$. If $\varphi : F \to G$ is a graded homomorphism of free modules, we denote by $I_k(\varphi)$ the ideal generated by the k-minors of a matrix representing φ. The same notation is also used if φ is a morphism of locally free sheaves.

Theorem 2.4. *Let $C \subset \mathbb{P}^n$ be a curve of degree at most $n-1$. Then the following conditions are equivalent:*

1. *C is an $(n-k)$–rope supported on the line L;*
2. *$I_C = ((I_L)^2, [x_0, \ldots, x_r]B)$ where the matrix B gives a map $\varphi_B : \oplus_{j=1}^k S(-\beta_j - 1) \to S^{r+1}(-1)$ with* $\mathrm{codim}(I_k(B)) = 2$;
3. *$I_C = ((I_L)^2, F_1, \ldots, F_k)$ where $V(F_1, \ldots, F_k) \subset \mathbb{P}^n$ is a scheme of codimension k which contains L and is smooth at the points of L.*

Proof. 1. $\Longrightarrow$ 2. According to Remark 2.2, the rope C corresponds to a rank $r+1-k$ subbundle E of the normal bundle $\mathcal{N}_L = \mathcal{O}_{\mathbb{P}^1}^{r+1}(1)$ via the exact sequence

$$0 \to E^* \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_L \to 0.$$

Let $E = \oplus_{i=0}^{r-k} \mathcal{O}_{\mathbb{P}^1}(1 - \alpha_i)$ where $\alpha_i \geq 0$. If we apply $\mathrm{Hom}(-, \mathcal{O}_{\mathbb{P}^1})$ to the embedding of E into $\mathcal{N}_L$, we get the exact sequence

$$(1) \qquad 0 \to \oplus_{j=1}^k \mathcal{O}_{\mathbb{P}^1}(-\beta_j - 1) \xrightarrow{\varphi_B} \mathcal{O}_{\mathbb{P}^1}^{r+1}(-1) \xrightarrow{\varphi_A} \oplus_{i=0}^{r-k} \mathcal{O}_{\mathbb{P}^1}(\alpha_i - 1) \to 0$$

where the ideals of maximal minors $I_k(\varphi_B)$ and $I_{r+1-k}(\varphi_A)$ have codimension 2 (cf., e.g., [9], Thm. 20.9, Prop. 18.2). Moreover, the sequence above provides the exact sequence of S-modules

$$0 \to \oplus_{j=1}^k S(-\beta_j - 1) \xrightarrow{\psi} I_L/(I_L)^2 \to I_L/I_C \to 0$$

where ψ is the composition $\oplus_{j=1}^k S(-\beta_j - 1) \xrightarrow{\varphi_B} S^{r+1}(-1) \xrightarrow{\cong} I_L/(I_L)^2$. Thus, $\mathrm{Im}\,\psi$ is generated by the residue classes of the entries of $[x_0, \ldots, x_r]B$. Since $I_C = (I_L)^2 + J$ for some ideal $J \subset I_L$, we can write J as $([x_0, \ldots, x_r]B)$ as claimed.

2. $\Longrightarrow$ 3. Now, we want to verify that $[x_0, \ldots, x_r]B$ is a regular sequence and defines a scheme which is smooth at every point $p \in L$. The Jacobian matrix J of $[x_0, \ldots, x_r]B$ is

$$J = [B^t, \frac{\partial B^t}{\partial t}[x_0 \ldots, x_r]^t, \frac{\partial B^t}{\partial u}[x_0, \ldots, x_r]^t].$$

Since the ideal $I_k(\varphi_B)$ of the maximal minors of B has codimension 2, B has rank k at every point not in $V(t,u)$, i.e. $V([x_0,\dots,x_r]B)$ is smooth on L.

3. $\Longrightarrow$ 1. This follows by [19], Corollary 1.7 and Proposition 1.9. $\square$

Remark 2.5. (i) If we apply $\mathrm{Hom}(-,\mathcal{O}_{\mathbb{P}^1}(-2))$ to the sequence (1) above, we obtain the sequence

$$0 \to \oplus_{i=0}^{r-k}\mathcal{O}_{\mathbb{P}^1}(-\alpha_i-1) \xrightarrow{\varphi_A^*} \mathcal{O}_{\mathbb{P}^1}^{r+1}(-1) \xrightarrow{\varphi_B^*} \oplus_{j=1}^{k}\mathcal{O}_{\mathbb{P}^1}(\beta_j-1) \to 0,$$

i.e., A^t is just the syzygy matrix of B^t. It follows that, via the sequence (1), there are two matrices A, B associated to the $(n-k)$–rope on L giving the morphisms $\varphi_A : \mathcal{O}_{\mathbb{P}^1}^{r+1}(-1) \to \oplus_{i=0}^{r-k}\mathcal{O}_{\mathbb{P}^1}(\alpha_i-1)$ and $\varphi_B : \oplus_{j=1}^{k}\mathcal{O}_{\mathbb{P}^1}(-\beta_j-1) \to \mathcal{O}_{\mathbb{P}^1}^{r+1}(-1)$ such that the ideals of the maximal minors of A, B have codimension 2 and satisfy $A\cdot B = 0$ (cf. also Proposition 2.9).

(ii) Throughout the paper we will use the notation of the last theorem and its proof. In particular, the matrices A and B will be used to describe properties of the rope C. Sometimes we will write A_C and B_C in order to stress the dependency of these matrices on the rope C.

(iii) We may order the forms F_i such that $\deg F_i = \beta_i + 1$.

(iv) The only n–rope supported on a line L is the first infinitesimal neighbourhood of L. If we take B as the zero matrix and A as the unit matrix of order $r+1$, we also get in this case the exact sequence (1) above and the same description of the homogeneous ideal as in the theorem.

Lemma 2.6. *We have* $\sum_{i=0}^{r-k}\alpha_i = \sum_{j=1}^{k}\beta_j$.

Proof. This is a standard consequence of the exact sequence (1) and the additivity of first Chern classes. $\square$

Now, we determine the Hilbert polynomial of an $(n-k)$–rope supported on L.

Proposition 2.7. *Let $I_C = ((I_L)^2, F_1,\dots,F_k)$ be the homogeneous ideal of an $(n-k)$-rope $C \subset \mathbb{P}^n$ supported on L and let β_i+1 be the degree of F_i. Then the Hilbert polynomial of C is*

$$P_C(z) = (n-k)z + 1 + \sum_{j=1}^{k}\beta_j.$$

Proof. This follows immediately by Theorem 2.4 and [19], Theorem 1.5. $\square$

Corollary 2.8. *Let $C \subset \mathbb{P}^n$ be a rope supported on L having degree d and (arithmetic) genus g. Then we have $g \le 0$. Furthermore, if C is non-degenerate then $g \le d-n$.*

Proof. C is degenerate if and only if $\beta_j = 0$ for some j. But if $k = n-d > -g = \sum_{j=1}^{k}\beta_j$, we must have that $\beta_j = 0$ for some j. $\square$

Now, we investigate when two different ideals define the same rope supported on the line L. To this end consider the two ideals $J_1 = ((I_L)^2, \underline{x}B_1)$ and $J_2 = ((I_L)^2, \underline{x}'B_2')$, where the entries of $\underline{x}$ and $\underline{x}'$ generate I_L, respectively, such that the two ideals define $(n-k)$–ropes supported on the line L. We may assume that $\underline{x} = [x_0\dots,x_r]$. Furthermore, after a base change in $\mathcal{O}_{\mathbb{P}^1}^{r+1}(-1)$, we can also assume $\underline{x} = \underline{x}'$ while replacing B_2' by $B_2 = FB_2'$ where $F \in GL_K(r+1)$. Then we have.

Proposition 2.9. *The ideals $J_1 = ((I_L)^2, \underline{x}B_1)$ and $J_2 = ((I_L)^2, \underline{x}B_2)$ define the same $(n-k)$–rope C if and only if $B_1 = B_2Q$ where the matrix Q gives a graded automorphism of $\oplus_{j=1}^{k}\mathcal{O}_{\mathbb{P}^1}(-\beta_j-1)$.*

Proof. We have that $J_1 = J_2$ if and only if B_1 and B_2 can be obtained from each other by elementary graded operations on the columns, i.e. if we have a graded automorphism Q of $\oplus_{j=1}^{k}\mathcal{O}_{\mathbb{P}^1}(-\beta_j - 1)$. □

As the last topic of this section, we present geometric descriptions of ropes supported on a line following ideas of Segre ([28]) and Harris ([14]), respectively.

Remark 2.10. (i) (*Ropes as a multiple structure on a scroll* (cf. [28]).) In Theorem 2.4, we associated a map $\varphi_A : \mathcal{O}_{\mathbb{P}^1}^{r+1}(-1) \to \oplus_{i=0}^{r-k}\mathcal{O}_{\mathbb{P}^1}(\alpha_i - 1)$ to every $(n-k)$–rope supported on a line $L \cong \mathbb{P}^1$. Each row A_i of A can be thought as a map $A_i : L \to \mathbb{P}^r$. Thus, these rows define $r+1-k$ rational (not necessarily normal) curves $\gamma_0, \ldots, \gamma_{r-k}$ in $\mathbb{P}^r$. According to Remark 2.5, we have that the points $A_0(t,u), \ldots, A_{r-k}(t,u)$ are linearly independent, for each $(t,u) \in \mathbb{P}^1$. Let $V \subseteq \mathbb{P}^r$ be the scroll which is the union of the $(r-k)$–planes $< A_0(t,u), \ldots, A_{r-k}(t,u) >$ spanned by the points $A_0(t,u), \ldots, A_{r-k}(t,u)$ where (t,u) varies in $\mathbb{P}^1$. Note that $\dim V = r+1-k$. Now, we embed $L \cong \mathbb{P}^1$ and $\mathbb{P}^r$ in $\mathbb{P}^{r+2} = \mathbb{P}^n$ in such a way that they are skew linear spaces. Hence, we can assume that $\mathbb{P}^1 = \mathrm{Proj}(K[t,u]), \mathbb{P}^r = \mathrm{Proj}(K[x_0, \ldots, x_r])$ and $\mathbb{P}^n = \mathrm{Proj}(K[x_0, \ldots, x_r, t, u])$. Let $X \subset \mathbb{P}^n$ be the scroll of dimension $r+2-k$ $(= n-k)$ which is the union of the linear spaces spanned by the points in L (corresponding to (t,u)) and $< A_0(t,u), \ldots, A_{r-k}(t,u) > \subset \mathbb{P}^r$. The homogeneous ideal of the scroll X is $I_X = I_V + ([x_0, \ldots, x_r]B)$ where B is the syzygy matrix of A. Then, an $(n-k)$–rope C is the first infinitesimal neighbourhood of L in X, i.e. $I_C = I_X + (I_L)^2$. But $I_X + (I_L)^2 = ((I_L)^2, [x_0, \ldots, x_r]B)$ because of the following result.

Claim. V is degenerate if and only if $\beta_j = 0$ for some j. In fact, V is contained in a hyperplane H if and only if $A_j(t,u) \in H$ for every $(t,u) \in L$, and for each $j = 0, \ldots, r-k$. If we set $H = V(\sum b_h x_h)$, the last condition becomes $\sum A_{jh}(t,u) b_h = 0$ for every $j = 0, \ldots, r-k$, i.e. A has a degree 0 syzygy and the linear form defining H is contained in the ideal $([x_0, \ldots, x_r]B)$.

(ii) (*Ropes as generalized double lines.*) In [14], Harris describes a double line in $\mathbb{P}^3$ by specifying the underlying line L and for each point $P \in L$ a normal direction to L by a map $L \to \mathbb{P}^3/L$. The discussion above shows that an $(n-k)$–rope in $\mathbb{P}^n$ is given by specifying the underlying line L and for each point $P \in L$ a linear space of normal directions to L having dimension k. This is done by giving a morphism

$$\varphi_B : \oplus_{j=1}^{k}\mathcal{O}_L(-\beta_j - 1) \to \mathcal{O}_L^{r+1}(-1)$$

which is injective at all points of L.

3. The Hartshorne-Rao module of the ropes

The goal of this section is to compute the Hartshorne-Rao module of a rope $C \subset \mathbb{P}^n$ supported on a line L. Then we single out the arithmetically Buchsbaum ropes.

Recall that there are two matrices A, B associated to each such rope whose entries are in $S = K[t,u] \cong R/I_L$ (cf. Remark 2.5). Whereas the matrix B is visible in the defining equations of C, the matrix A determines its Hartshorne-Rao module.

Proposition 3.1. *Let $C \subset \mathbb{P}^n$ be an $(n-k)$–rope supported on the line L with homogeneous ideal $I_C = ((I_L)^2, [x_0, \ldots, x_r]B)$. Then its Hartshorne-Rao module $H^1_*(\mathcal{I}_C)$ is isomorphic as R–module to* $\mathrm{coker}(S^{r+1}(-1) \xrightarrow{\varphi_A} \oplus_{j=0}^{r-k} S(\alpha_j - 1))$.

Proof. If $k = 0$ then C is arithmetically Cohen-Macaulay. Thus, the result is true in this case (cf. Remark 2.5(iv)).

Assume $k > 0$. We have the following exact sequences of R-modules

$$0 \to \oplus_{j=1}^{k} R/I_L(-\beta_j - 1) \xrightarrow{\varphi_B} (R/I_L)^{r+1}(-1) \xrightarrow{\varphi_A} \oplus_{i=0}^{r-k} R/I_L(\alpha_i - 1) \to \operatorname{coker}\varphi_A \to 0$$

and

$$0 \to \oplus_{j=1}^{k} R/I_L(-\beta_j - 1) \xrightarrow{\varphi_B} (R/I_L)^{r+1}(-1) \to I_L/I_C \to 0$$

(cf. the proof of Theorem 2.4). Note, that the Koszul resolution of R/I_L provides $\operatorname{Ext}_R^{n-1}(R/I_L, R) \cong R/I_L(n-1)$. Thus, dualizing the second sequence with respect to R we obtain the exact sequence

$$\ldots \to (R/I_L)^{r+1}(n) \xrightarrow{\varphi_B^*(n-1)} \oplus_{j=1}^{k} R/I_L(\beta_j + n) \to \operatorname{Ext}_R^{n-1}(I_L/I_C, R) \to 0.$$

It implies $\operatorname{Ext}_R^{n-1}(I_L/I_C, R) \cong \operatorname{coker}\varphi_B^*(n-1)$.

Dualizing the first sequence with respect to $S \cong R/I_L$ we get by local duality

$$\operatorname{coker}\varphi_A \cong \operatorname{Ext}_S^2(\operatorname{coker}\varphi_B^*, S) \cong (\operatorname{coker}\varphi_B^*)^\vee(2)$$

where $M^\vee$ denotes the graded K-dual $\operatorname{Hom}_K(M, K)$ of the graded R-module M. Using local duality again, we obtain

$$H^1_{\mathfrak{m}}(I_L/I_C) \cong \operatorname{Ext}_R^{n-1}(I_L/I_C, R)^\vee(n+1) \cong \operatorname{coker}\varphi_A.$$

Finally, since L is arithmetically Cohen-Macaulay, the exact sequence

$$0 \to I_L/I_C \to R/I_C \to R/I_L \to 0$$

implies

$$H^1_*(\mathcal{I}_C) \cong H^1_{\mathfrak{m}}(R/I_C) \cong H^1_{\mathfrak{m}}(I_L/I_C).$$

Now, our claim follows. □

Corollary 3.2. *The Rao function $\rho_C(i) := h^1(\mathcal{I}_C(i))$ of an $(n-k)$–rope $C \subset \mathbb{P}^n$ supported on L is*

$$\rho_C(i) = \sum_{j=0}^{r-k} \binom{i+\alpha_j}{1} + \sum_{j=1}^{k} \binom{i-\beta_j}{1} - (r+1)\binom{i}{1}.$$

Proof. This follows directly from the exact sequence (1) and Proposition 3.1. □

The Hartshorne–Rao module of double lines is particularly simple.

Corollary 3.3. *Let $C \subset \mathbb{P}^n$ be a double line. Then its Hartshorne–Rao module is generated by one element.*

Proof. For a double line C we have $n-k = 2$, thus $r-k = 0$. Hence, the Hartshorne-Rao module of C is isomorphic to $S(\alpha_0 - 1)/(A)$ where (A) is the ideal generated by the entries of the matrix A. □

Finally, we investigate properties of ropes have with a given Hartshorne-Rao module. From now on it is understood that the ropes we are studying are always supported on a line.

Corollary 3.4. *Let M be a graded R–module of finite length. Then the arithmetic genus of a rope $C \subset \mathbb{P}^n$, whose Hartshorne-Rao module is isomorphic to M as R–module, depends only on M.*

Furthermore, if in addition C is non-degenerate, then also the degree of C is uniquely determined by M.

Proof. According to Proposition 3.1, a minimal free resolution of the Hartshorne-Rao module $M(C)$ as S-module can be obtained from the sequence (1) by cancelling redundant free summands. It follows that the shifts ≤ -2 in the last free module in the associated sequence (1) to C are uniquely determined by the isomorphism class of $M(C)$. Hence, the first claim follows by Proposition 2.7.

If C is non-degenerate, then all the entries of the matrix B_C are positive by Theorem 2.4. Thus, the shifts in the last free module in the associated sequence (1) to C agree with the ones of the last free module of the minimal free resolution of M. Since $M(C) \cong M$ it follows that the size of the matrix B_C is determined by M. Therefore, Theorem 2.4 shows the second claim. □

Remark 3.5. Even if the rope C is possibly degenerate there is some information on the degree of C by knowing its Hartshorne-Rao module. In fact, let

$$0 \to F_2 \to F_1 \to F_0 \to M \to 0$$

be a minimal free resolution of M as S–module. Then there is the following estimate for the degree of C:

$$1 + \operatorname{rank}_S F_0 \leq \deg C \leq n - \operatorname{rank}_S F_2.$$

Proposition 3.1 shows that the Hartshorne-Rao module of a rope in $\mathbb{P}^n$ is annihilated by at least $n-1$ linearly independent linear forms. The next result describes when there are more such linear forms.

Proposition 3.6. *Let $C \subset \mathbb{P}^n$ be a rope which is not arithmetically Cohen-Macaulay. Then the following conditions are equivalent:*

1. *There are at least n linearly independent linear forms annihilating the Hartshorne-Rao module $M(C)$ of C;*
2. *C is arithmetically Buchsbaum and $M(C) \cong K^m$ for some positive integer $m \leq \frac{n-1}{2}$.*

Moreover, in this case C has genus $-m$ and is cut out by quadrics and possibly linear forms. If C is non-degenerate then C has degree $n-m$.

Proof. 1. $\Longrightarrow$ 2. Let l be a linear form annihilating $M(C)$, but not being contained in I_L. Then, $M(C) \cong \operatorname{coker} \varphi_A$ is an S/lS-module. Therefore, its minimal free resolution as module over $T := S/lS$ has the form

$$0 \to \oplus_{j=1}^m T(-\delta_j) \to \oplus_{i=1}^m T(-\gamma_i) \to M(C) \to 0$$

for some positive integer m. Thus, the minimal free resolution as S-module has the shape

$$0 \to \oplus_{j=1}^m S(-\delta_j - 1) \to \oplus_{j=1}^m S(-\delta_j) \bigoplus \oplus_{i=1}^m S(-\gamma_i - 1) \to \oplus_{i=1}^m S(-\gamma_i) \to M(C) \to 0.$$

The exact sequence (1) provides the following free resolution of $M(C)$

$$0 \to \oplus_{j=1}^k S(-\beta_j - 1) \to S^{r+1}(-1) \to \oplus_{i=0}^{r-k} S(\alpha_i - 1) \to M(C) \to 0.$$

The comparison of the two resolutions shows $\delta_j = 1$, $\gamma_i = 0$ for all i and j, and $2m \leq r+1 = n-1$. Since T is a polynomial ring in one variable it follows that $M(C) \cong K^m$ as claimed.

2. $\Longrightarrow$ 1. is trivial.

In order to show the final claim, note that we may assume $\beta_1 = \ldots = \beta_m = 1$ and $\beta_{m+1} = \ldots = \beta_k = 0$. But the genus of C is $-m = -\beta_1 - \ldots - \beta_k$ (cf. Corollary 2.7). Moreover, Theorem 2.4 shows that I_C is generated by m quadrics outside of $(I_L)^2$ and $k-m$ linear forms if C has degree $n-k$. This completes the proof. □

Observe that not all non-degenerate ropes in $\mathbb{P}^n$, which are cut out by quadrics and satisfy $d-g=n$, are arithmetically Buchsbaum. For example, for any $n \geq 4$ there is a non-degenerate rope of degree $n-2$ and genus 2 which is cut out by quadrics, but whose Hartshorne-Rao module is isomorphic to $S/(t,u)^2$.

Corollary 3.7. *Let $C \subset \mathbb{P}^n$ be a rope with genus $g < -\frac{n-1}{2}$. Then C is not arithmetically Buchsbaum.*

Proof. This is an immediate consequence of Proposition 3.6. □

Later on, we will need the following result.

Corollary 3.8. *Let $C, E \subset \mathbb{P}^n$ be $(n-k)$–ropes. If C is not arithmetically Buchsbaum, then the Hartshorne-Rao modules of C and E are isomorphic as R–modules if and only if C and E are supported on the same line L and $A_E = QA_CP$ where Q is a matrix providing a graded automorphism of $\oplus_{i=0}^{r-k} S(\alpha_i - 1)$ and $P \in GL_K(n-1)$.*

Proof. Proposition 3.6 guarantees that the linear forms in R annihilating $M(C) \cong M(E)$ generate the ideal of a line L. Thus, Proposition 3.1 shows that L must be the support of C and E.

Since C and E have the same degree, the matrices A_C and A_E have the same size. Hence, the second claim follows again by Proposition 3.1. □

4. Families of arithmetically Gorenstein curves

In this section, we describe a family $\mathcal{F}_n$ of arithmetically Gorenstein curves in $\mathbb{P}^n$ of degree $n+1$ supported on a fixed line for each $n \geq 3$ with the property that every rope in $\mathbb{P}^n$ of degree d supported on the line L can be linked by means of every curve in the family $\mathcal{F}_n$ to a rope supported on L having degree $n+1-d$. The family $\mathcal{F}_n$ depends on the choice of the supporting line. As in the previous sections, we will fix the supporting line L defined by $x_0 = \ldots = x_{n-2} = 0$, i.e. $L = \operatorname{Proj} S$ where $S = K[t,u]$. In order to achieve the goals just described, we have to require that each curve $X \in \mathcal{F}_n$ satisfies the following conditions:

(i) X is supported on the line L;
(ii) $\deg(X) = n+1$;
(iii) the tangent space $T_{x,X}$ to X at x has dimension n for every point $x \in L$.

In fact, the first condition is clear, because the ropes are supported on the line L. The second requirement stems from the fact that the Hartshorne-Rao modules of linked ropes are dual to each other. Thus, comparing with the sequence (1) we see that the sum of the degrees of the two ropes should be $n+1$. To explain the third condition, we notice that a curve X can be used to link a curve C to another curve D if for each point $x \in C$ we have $T_{x,C} \subseteq T_{x,X}$. But, we want to use a curve $X \in \mathcal{F}_n$ to link whatever rope C we choose. Hence, we have to require that the tangent space $T_{x,X}$ has the maximal dimension n for every $x \in L$. As starting point for the construction of the family $\mathcal{F}_n$, we compute the generic initial ideal of the curves in the family with respect to the reverse lexicographic order. For the theory of generic initial ideals we refer to [11] and [9].

Proposition 4.1. *Let $X \subset \mathbb{P}^n$ be a non-degenerate arithmetically Cohen-Macaulay curve of degree $n+1$. Assume that the characteristic of the ground field K is not two. Then the generic initial ideal $\operatorname{gin}(I_X)$ of X is generated by the monomials x_r^3 and x_ix_j with $0 \leq i,j \leq r, (i,j) \neq (r,r)$.*

Proof. Let Γ be the general hyperplane section of X. Since X is arithmetically Cohen-Macaulay, $\operatorname{gin} I_X$ and $\operatorname{gin} I_\Gamma$ are generated by the same monomials. Moreover, $\Gamma \subset \mathbb{P}^{n-1}$ is non-degenerate. Thus, its Hilbert function is $H_\Gamma = (1, n, n+1, n+1, \dots)$. It follows that $\operatorname{rank}_K (I_\Gamma)_2 = \binom{r}{2} - 1$. Since $\operatorname{gin} I_\Gamma$ is a stable ideal, it must contain the largest $\binom{r}{2} - 1$ monomials of degree 2, i.e. $\operatorname{gin}(I_\Gamma)$ contains the monomials $x_i x_j$ with $0 \leq i, j \leq r, (i,j) \neq (r,r)$. (For this argument we need the assumption on the characteristic.) Let J denote the ideal generated by these monomials. Then we get $\operatorname{rank}_K(J)_3 = \operatorname{rank}_K(I_\Gamma)_3 - 1$. Therefore, $\operatorname{gin} I_\Gamma$ has exactly one minimal generator of degree 3. By stability it must be x_r^3. Since $J + x_r^3$ has the same Hilbert function as I_Γ, both ideals are equal proving the claim. □

We now use this result to determine all arithmetically Gorenstein curves $X \subset \mathbb{P}^n$ satisfying the conditions (i) - (iii) above. Condition (i) yields that $I_X \subset I_L$. The proposition above shows that I_X is generated by polynomials of the form

$$f_{ij} = x_i x_j - h_{ij} x_r^2 + x_0 l_{ij}^0 + \cdots + x_r l_{ij}^r$$

and

$$f = x_r^3 + q_0 x_0 + \ldots + q_r x_r$$

where $h_{ij} \in K$, $l_{ij}^h \in (S)_1$ for every i, j, h and $q_0, \dots, q_r \in (S)_2$. Then, the Jacobian matrix of I_X contains the forms $l_{i,j}^h$ and q_k as entries. But it must be the zero matrix by condition (iii). It follows that I_X is generated by $\{x_i x_j - h_{ij} x_r^2, x_r^3 \mid 0 \leq i \leq j \leq r, (i,j) \neq (r,r)\}$. The Gorenstein ideals among the ideals of this shape are characterized in the next result. It does not depend on the characteristic of K.

Theorem 4.2. *Let $I_X \subset R$ be the ideal generated by x_r^3 and $x_i x_j - h_{ij} x_r^2$ where $h_{ij} \in K$ and $0 \leq i \leq j \leq r, (i,j) \neq (r,r)$ and let $G = [g_{ij}]$ be the symmetric $(r+1) \times (r+1)$ matrix defined by*

$$g_{ij} = \begin{cases} h_{ij} & \text{if} \quad i \leq j, (i,j) \neq (r,r) \\ h_{ji} & \text{if} \quad i \geq j, (i,j) \neq (r,r) \\ 1 & \text{if} \qquad i = j = r \end{cases}$$

Then, I_X is the ideal of an arithmetically Gorenstein curve if and only if the matrix G is non-singular.

Furthermore, in this case x_r^3 is not a minimal generator of I_X.

Proof. Put $T := K[x_0, \dots, x_n] \cong R/(t,u)R$. Then the ring $\overline{R} = T/I_X T$ is an artinian reduction of the ring R/I_X. Let $\mathfrak{m}$ be the maximal homogeneous ideal of the ring $\overline{R}$. Of course, $\mathfrak{m}$ is generated by $x_0, \dots, x_r$. Since $\mathfrak{m}^3 = 0$, the ideal I_X is Gorenstein if and only if the multiplication map:

$$\phi : \mathfrak{m}/\mathfrak{m}^2 \times \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}^2$$

is non-degenerate (cf. [2]). By the definition of I_X, $\mathfrak{m}/\mathfrak{m}^2$ is generated by $\{x_0, \dots, x_r\}$ and $\mathfrak{m}^2$ is generated by x_r^2. Hence, we obtain:

$$\phi(x_i, x_j) = g_{ij} x_r^2.$$

This means that the matrix associated to the map ϕ is exactly the matrix G. The conclusion follows. □

Remark 4.3. Let $\underline{x}$ denote the vector $[x_0, \dots, x_r]$. The matrix $x_r^2 G$ can be obtained from the product $\underline{x}^t \cdot \underline{x}$ by reduction modulo I_X, where, as usual, $\underline{x}^t$ is the transpose of the matrix $\underline{x}$.

Definition 4.4. We denote by $\mathcal{F}_n$ the set of arithmetically Gorenstein curves $X \subset \mathbb{P}^n$ whose homogeneous ideal I_X is generated by $x_i x_j - h_{ij} x_r^2$ where $h_{ij} \in K$ and $0 \le i \le j \le r, (i,j) \neq (r,r)$.

We think of $\mathcal{F}_n$ as a family of curves depending on the field elements h_{ij}. The curves in $\mathcal{F}_4$ are arithmetically Gorenstein of codimension 3. Hence, their homogeneous ideals are generated by the Pfaffians of a skew-symmetric matrix (cf. [4]). For completeness, we compute this matrix.

Proposition 4.5. *Up to a change of coordinates, the minimal free resolution of the homogeneous ideal I_X of a curve $X \in \mathcal{F}_4$ is*

$$0 \to R(-5) \to R^5(-3) \xrightarrow{E} R^5(-2) \to I_X \to 0$$

where the skew-symmetric matrix E is

$$E = \begin{pmatrix} 0 & 0 & cy_2 & y_1 & by_2 \\ 0 & 0 & y_1 & y_2 & y_0 \\ -cy_2 & -y_1 & 0 & y_0 & ay_2 \\ -y_1 & -y_2 & -y_0 & 0 & 0 \\ -by_2 & -y_0 & -ay_2 & 0 & 0 \end{pmatrix}$$

and $y_0 = x_0 - h_{02}x_2, y_1 = x_1 - h_{12}x_2, y_2 = x_2$, and a, b, c, are suitable elements in K depending on h_{ij}'s.

Proof. First, we explicitly write the generators of the ideal I_X as in Proposition 4.2:

$$f_1 = x_0^2 - h_{00}x_2^2; f_2 = x_0x_1 - h_{01}x_2^2; f_3 = x_1^2 - h_{11}x_2^2;$$
$$f_4 = x_0x_2 - h_{02}x_2^2; f_5 = x_1x_2 - h_{12}x_2^2; f_6 = x_2^3.$$

Second, after performing the following change of coordinates

$$y_0 = x_0 - h_{02}x_2; y_1 = x_1 - h_{12}x_2; y_2 = x_2; t' = t; u' = u$$

we can choose as new set of generators of the ideal I_X

$$f_1 - 2h_{02}f_4 = y_0^2 - ay_2^2; f_2 - h_{12}f_4 - h_{02}f_5 = y_0y_1 - by_2^2;$$
$$f_3 - 2h_{12}f_5 = y_1^2 - cy_2^2; f_4 = y_0y_2; f_5 = y_1y_2; f_6 = y_2^3$$

where $a = h_{00} - h_{02}^2, b = h_{01} - h_{02}h_{12}, c = h_{11} - h_{12}^2$. The determinant of the matrix associated to the new set of generators of I_X (in the sense of Proposition 4.2) is $ac-b^2 \neq 0$. Moreover, f_6 is not a minimal generator for I_X, because $y_2f_1 - y_0f_4 + af_6 = 0$ and $y_2f_2 - y_1f_4 + bf_6 = 0$, but a and b cannot be both equal to zero. Now, the claim follows from the computation of the Pfaffians of the matrix E. $\square$

5. Even Gorenstein liaison classes of ropes

In this section we will combine all the previous results in order to study ropes from a liaison theoretic point of view.

Let $C \subset \mathbb{P}^n$ be an $(n-k)$–rope supported on the line L and defined by the ideal $I_C = ((I_L)^2, F_1, \dots, F_k) = ((I_L)^2, \underline{x}B_C)$ and let $X \in \mathcal{F}_n$ be an arithmetically Gorenstein curve supported on L with homogeneous ideal I_X and associated matrix G (cf. Proposition 4.2). We have that $I_X \subseteq (I_L)^2 \subseteq I_C$. Thus, we can Gorenstein link C via X. We want to determine the residual curve D. The first step is.

Lemma 5.1. *Let $D \subset \mathbb{P}^n$ be the curve linked to the $(n-k)$–rope C via $X \in \mathcal{F}_n$. Then D is a $(k+1)$–rope.*

Proof. By definition we have that $I_D = I_X : I_C$. Notice that $(I_L)^2 \subseteq I_D$ because $(I_L)^3 \subseteq I_X$ and $F_i \cdot (I_L)^2 \subseteq (I_L)^3$, for every $i = 1, \dots, k$. It follows by Proposition 2.3 that D is a $(k+1)$–rope. □

According to Theorem 2.4, we can write the homogeneous ideal of D as $I_D = ((I_L)^2, \underline{x}B_D)$ for a suitable matrix B_D. Now, we want to relate the matrix B_D to the matrix G associated to X and to the matrix A_C (cf. Remark 2.5).

Proposition 5.2. *Let C (resp. D) be an $(n-k)$ – rope (resp. $(k+1)$ – rope) with ideal $I_C = ((I_L)^2, \underline{x}B_C)$ (resp. $I_D = ((I_L)^2, \underline{x}B_D)$) supported on L. Then C and D are directly linked by a Gorenstein curve X of the family $\mathcal{F}_n$ if and only if $B_D = G^{-1}A_C^tP$, where G is the matrix associated to X, the matrix P provides a graded automorphism of $\oplus_{i=0}^{r-k} S(\alpha_i - 1)$ and $S = K[t,u]$.*

Proof. Let us denote the i-th column of a matrix M by $(M)_i$. Then, using Theorem 2.4, we obtain from the definition of linked schemes

$$(\underline{x}(B_C)_i)(\underline{x}(B_D)_j) = 0 \ (\mathrm{mod} I_X) \qquad \text{for all } i, j.$$

We can write all these equations as

$$(\underline{x}B_C)^t(\underline{x}B_D) = 0 \ (\mathrm{mod} I_X).$$

It follows by Remark 4.3 that

$$(B_C)^t(\underline{x})^t\underline{x}B_D = (B_C)^t x_r^2 G B_D = 0 \ (\mathrm{mod} I_X).$$

Since the entries of B_C, B_D, G are in S we get

$$(B_C)^t G B_D = 0.$$

This shows that the columns of GB_D are syzygies of the module generated by the columns of $(B_C)^t$. Since the ideal of maximal minors of GB_D has codimension 2, the columns of GB_D and $(A_C)^t$ generate isomorphic S-modules. This means, that there is a graded automorphism of $\oplus_{i=0}^{r-k} S(\alpha_i - 1)$ given by a matrix P such that $(GB_D)^t = P^t A_C$, proving our claim. □

As a consequence, we get a relation between two ropes C and E which are bilinked.

Corollary 5.3. *Two $(n-k)$–ropes C and E in $\mathbb{P}^n$ are linked in two steps by two arithmetically Gorenstein curves X_1 and X_2 of the family $\mathcal{F}_n$ if and only if $A_E = QA_CG_1^{-1}G_2$, where G_i is the matrix associated to X_i, $i = 1, 2$, and Q provides a graded automorphism of $\oplus_{i=0}^{r-k} S(\alpha_i - 1)$.*

Proof. By assumption, there is a $(k+1)$–rope D such that C and D are directly linked by $X_1 \in \mathcal{F}_n$ and E and D are directly linked by $X_2 \in \mathcal{F}_n$. According to Proposition 5.2, we have that $B_D = G_1^{-1}(A_C)^tP_1 = G_2^{-1}(A_E)^tP_2$. The claim follows. □

Now, we can prove our main result.

Theorem 5.4. *Let $C, E \subset \mathbb{P}^n$ be $(n-k)$–ropes which are not arithmetically Buchsbaum. Then, the following conditions are equivalent:*

1. *C and E are in the same even Gorenstein liaison class;*
2. *the Hartshorne-Rao modules $M(C)$ and $M(E)$ of C and E are isomorphic as R–modules.*

Proof. It is a general fact of liaison theory that 1. implies 2. (cf., e.g., [22], Corollary 5.13). Note, that in our special situation no degree shift is needed. Thus, it suffices to show the converse.

Our assumption ensures that Corollary 3.8 applies. It shows that C and E are supported on the same line L and $A_E = QA_C P$ where Q is a matrix providing a graded automorphism of $\oplus_{i=0}^{r-k} S(\alpha_i - 1)$ and $P \in GL_K(n-1)$. Hence, using Corollary 5.3 successively our assertion follows from the following

Claim: *$GL_K(n-1)$ is equal to its subgroup H generated by $\{G_1^{-1}G_2 \mid G_i$ is associated to an arithmetically Gorenstein curve $X_i \in \mathcal{F}_n$* (cf. Theorem 4.2) $\}$

Every matrix $P \in GL_K(n-1)$ is the product of invertible matrices describing elementary column operations. Thus, it is enough to prove that each such elementary matrix is in H. By considering block diagonal matrices we see that it suffices to show the latter statement for 2×2 matrices.

There are the following three types of 2×2 elementary matrices

$$M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \ M_2 = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}; \ M_3 = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

where we may assume that $x \neq 0$. Moreover, it is clear that the identity matrix is in H. Hence, all matrices of the form $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ where $x \neq 0$ belong to H as well (cf. Theorem 4.2). Therefore, our claim follows from the following identities:

$$M_1 = \begin{pmatrix} a & b \\ b & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & a+b-b^2 \\ a+b-b^2 & 1 \end{pmatrix} \begin{pmatrix} a & 1-b \\ 1-b & 1 \end{pmatrix}^{-1};$$

$$M_2 = M_1 \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} M_1;$$

$$M_3 = \begin{pmatrix} a & b \\ b & 1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{b(a+xb)}{b+x} & b \\ b & 1 \end{pmatrix} \begin{pmatrix} \frac{b}{b+x} & 0 \\ 0 & 1 \end{pmatrix}^{-1}.$$

□

Remark 5.5. (i) Let C be a rope which is not arithmetically Buchsbaum. Then the previous result characterizes all the ropes having degree $\deg C$ which are in the even liaison class of C.

(ii) It is shown in [22], Theorem 3.10 that each even liaison class of codimension c subschemes determines a stable equivalence class of certain $(c+1)$–syzygies. In codimension two this map is in fact injective and provides the so-called Rao's correspondence. It is not known if the same result is true in codimension ≥ 3. For curves the hoped-for Rao's correspondence is equivalent to the statement that two curves are evenly linked if and only if their Hartshorne-Rao modules are isomorphic. Our Theorem 5.4 points towards that direction.

Remark 5.6. Some of the methods developed here have been extended in our subsequent paper [24] in order to extend Theorem 5.4 in two directions. First, the assumption in Theorem 5.4 that the curves C and E have the same degree can be dropped (cf. [24], Theorem 5.6). Second, we can allow arithmetically Buchsbaum curves with 1-dimensional Hartshorne-Rao module (cf. [24], Proposition 4.7). As a consequence, two ropes in $\mathbb{P}^4$ are in the same even Gorenstein liaison class if and only if their Hartshorne-Rao modules are isomorphic. It remains open if this statement is also true for arithmetically Buchsbaum curves in $\mathbb{P}^n$ with $n \geq 5$ though [24], Proposition 4.10 reduces the number of cases to be checked.

We are now ready for the proof of the theorem stated in the introduction.

Proof of Theorem 1.1. The assumption on the genus guarantees that the ropes are not arithmetically Buchsbaum by Corollary 3.7. Thus, it only remains to show the equivalence of the conditions (b) and (c).

Let us first assume that $M(C) \cong M(E)$. Then condition 2. follows by Corollary 3.4. Furthermore, Corollary 3.8 shows condition 1. and $A_E = QA_CP$ where Q is a matrix providing a graded automorphism of $\oplus_{i=0}^{r-k} S(\alpha_i - 1)$ and $P \in GL_K(n-1)$. Since B_C and B_E are syzygy matrices of A_C and A_E, respectively, we can assume by Proposition 2.9 that $B_E = P^{-1} \cdot B_C$, i.e. condition 3. is satisfied.

Second, we show that conditions 1. - 3. imply $M(C) \cong M(E)$. By Proposition 3.1, the latter is equivalent to $\operatorname{coker} \varphi_{A_C} \cong \operatorname{coker} \varphi_{A_E}$. But this follows from the exact sequence (1) (cf. also Remark 2.5). □

For double lines we can improve Theorem 5.4.

Theorem 5.7. *Let $C, E \subset \mathbb{P}^n$ be two curves of degree two. Then C and E belong to the same even liaison class if and only if their Hartshorne-Rao modules are isomorphic.*

Proof. Again, it suffices to show the sufficiency of the condition on the Hartshorne-Rao modules for being evenly linked.

If C and E are arithmetically Cohen-Macaulay they are complete intersections, thus the result is known.

If C and E are not arithmetically Buchsbaum, Corollary 3.8 applies. Thus, the proof of Theorem 5.4 shows that C and E belong to the same even liaison class.

Finally, assume that C and E are arithmetically Buchsbaum. Then Proposition 3.6 implies that $M(C) \cong M(E) \cong K$. Thus, after a change of coordinates we may assume that the homogeneous ideal of C is $I_C = (x_0^2, x_0x_1, x_1^2, x_2, \ldots, x_r, x_0t - x_1u)$. Using the scheme defined by $(x_2, \ldots, x_r, x_0x_1, x_0t - x_1u)$ we can link C to the pair of skew lines defined by $(x_0, x_2, \ldots, x_r, u) \cap (x_1, x_2, \ldots, x_r, t)$. Similarly, we can link E to a pair of skew lines. Our claim follows because any two pairs of two skew lines are evenly linked by [27] for $n = 3$ and by [6], [13] or [16] if $n \geq 4$. □

Together with Proposition 3.1 this result gives a complete description of the even liaison classes containing a curve of degree two.

6. Construction of quasi-extremal curves

In [8] an optimal bound for the Rao function of a curve with non-degenerate general hyperplane section was established. The curves attaining this bound in non-negative degrees were called quasi-extremal curves and investigated in [26]. The goal of this section is to show that such curves can be constructed as the union of an $(n-1)$–rope and a suitable plane curve.

We begin by recalling the precise definition and some results.

Definition 6.1. Let $C \subset \mathbb{P}^n$ be a curve of degree $d \geq n+1$ and genus g, and put $a = \binom{d-n+1}{2} - g$. Then C is said to be a *quasi-extremal curve* if its general hyperplane section is non degenerate and

$$h^1(\mathcal{I}_C(j)) = \begin{cases} a & \text{if} \quad 0 \leq j \leq d-n+1 \\ \binom{d-n+2}{2} - g - j + 1 & \text{if} \quad d-n+1 \leq j \leq \binom{d-n+2}{2} - g \\ 0 & \text{if} \quad j \geq \binom{d-n+2}{2} - g. \end{cases}$$

While the Rao function of a quasi-extremal curve is fixed in non-negative degrees, it can vary in negative degrees. In fact, in [26] it is shown:

Theorem 6.2. *A numerical function is the Rao function of a quasi-extremal curve if and only if its graph in negative degrees is a polygonal whose edges have an increasing slope, not exceeding $n-2$ and passing through the point $(0, a)$.*

Some insight in the geometry of such curves is given by the following result in [26]:

Proposition 6.3. *Let C be a quasi-extremal curve of degree d. Then C contains a plane curve of degree $d-n+2$.*

It follows that a quasi-extremal curve with negative genus must have a non-reduced structure on its part which does not lie in the plane. Nevertheless, the next result indicates that we have to expect that this plane must contain the support of the non-reduced part.

Lemma 6.4. *Let $C \subset \mathbb{P}^n$ be the union of a rope D supported on a line L and a plane curve P of degree ≥ 3 contained in the plane π. Assume that the quadrics containing D cut out the first infinitesimal neighbourhood of L. Then we have: if C is a quasi-extremal curve, then L is contained in π.*

Proof. The assumptions imply that $(I_C)_2 = (I_\pi)_2 \cap ((I_L)^2)_2$. Let J denote the ideal generated by $(I_C)_2$ and consider the following exact sequence

$$0 \to J \to I_\pi \oplus (I_L)^2 \to I_\pi + (I_L)^2 \to 0.$$

According to [26], Lemma 1.5, the Hilbert polynomial of J is $p_J(z) = \binom{z+n}{n} - \frac{z^2+(2n-1)z+2}{2}$. Thus, the Hilbert polynomial of the ideal $(I_L)^2 + I_\pi$ is $p(z) = \binom{z+n}{n} - (2z+1)$, i.e. the ideal defines a curve. Hence, $L \subseteq \pi$. $\square$

Thus, if we want to construct a quasi-extremal curve as in the lemma, we have to choose D as an $(n-1)$–rope with support L in π and P as a plane curve in π of degree $d-n+1$. Thus, we may write $I_P = (x_0, \ldots, x_{r-1}, G)$, where $\deg(G) = d-n+1$. The ideal of D has the form $I_D = ((I_L)^2, F)$ where $F = \sum_{i=0}^r x_i f_i$ has degree β. It follows that $I_D + I_P = (x_0, \ldots, x_{r-1}, x_r^2, x_r f_r, G)$. Hence, the intersection of C and D is a 0-dimensional scheme of degree $d-n+1$ if and only if $x_0, \ldots, x_r, f_r, G$ is a regular sequence.

Using this notation we can state the main result of this section.

Theorem 6.5. *If the rope D and the plane curve P meet in a 0-dimensional scheme of degree $d-n+1$, then the union of D and P is a quasi-extremal curve of degree d and genus $\binom{d-n+1}{2} - \beta$ where $-\beta$ is the genus of D.*

Proof. Put $C = D \cup P$ and consider the exact sequence

$$0 \to I_C \to I_D \oplus I_P \to I_D + I_P \to 0.$$

It implies for the Hilbert polynomial of C

$$p_C = dz + 1 - \binom{d-n+1}{2} + \beta$$

and the associated cohomology sequence

$$(*) \qquad 0 \to \frac{(I_D + I_P)^{\mathrm{sat}}}{(I_D + I_P)} \to H^1_*(\mathcal{I}_C) \to H^1_*(\mathcal{I}_D) \to H^1_*(\mathcal{I}_{D \cap P}) \to \ldots.$$

It follows that C is a locally Cohen-Macaulay curve of degree d and genus $g = \binom{d-n+1}{2} - \beta$.

According to our assumption, we have $I_{D\cap P} = (I_D + I_P)^{\text{sat}} = (x_0, \ldots, x_r, G)$. Thus, the natural inclusion $(I_L)^2 \hookrightarrow I_{D\cap P}$ factors through I_L. Hence, the map $H^1_*(\mathcal{I}_D) \to H^1_*(\mathcal{I}_{D\cap P})$ in the sequence $(*)$ factors through $H^1_*(\mathcal{I}_L) = 0$, i.e. it must be the zero map.

Therefore, we obtain the short exact sequence

$$0 \to \frac{I_{D\cap P}}{I_D + I_P} \to H^1_*(\mathcal{I}_C) \to H^1_*(\mathcal{I}_D) \to 0. \tag{2}$$

Using the assumption on $D \cap P$ again, straightforward calculations give the Hilbert function of $I_D + I_P$. It follows

$$\dim_K \left(\frac{I_{D\cap P}}{I_D + I_P}\right)_j = \begin{cases} 0 & \text{if} \quad j \le 0 \\ j & \text{if} \quad 0 \le j \le d-n+1 \\ d-n+1 & \text{if} \quad d-n+1 \le j \le \beta \\ d+\beta-n+1-j & \text{if} \quad \beta \le j \le d+\beta-n+1 \\ 0 & \text{if} \quad j \ge d+\beta-n+1 \end{cases}$$

Applying Corollary 3.2 to the $(n-1)$–rope D we get from the sequence (2)

$$h^1(\mathcal{I}_C(j)) = \begin{cases} h^1(\mathcal{I}_D(j)) & \text{if} \quad j \le 0 \\ \beta & \text{if} \quad 0 \le j \le d-n+1 \\ d+\beta-n+1-j & \text{if} \quad d-n+1 \le j \le d+\beta-n+1 \\ 0 & \text{if} \quad j \ge d+\beta-n+1. \end{cases}$$

It is easy to see that the general hyperplane section of C is non-degenerate. Hence, C is a quasi-extremal curve. □

Remark 6.6. The homogeneous ideal of the quasi-extremal curve C in the theorem is $I_C = I_D \cap I_P = (x_0^2,\ x_0x_1, \ldots, x_{r-1}x_r, x_r^2G, FG)$. Ideals of this form were studied algebraically in [26], Lemma 3.3.

The genus g of a quasi-extremal curve C of degree d satisfies $g \le \binom{d-n+1}{2}$. Moreover, if $g = \binom{d-n+1}{2}$ then C is arithmetically Cohen-Macaulay (cf. [8], Corollary 2.2), i.e. the Rao function vanishes. Such curves of maximal genus were constructed in [8], Proposition 2.4. Examples of quasi-extremal curves of every possible non-vanishing Rao function can be constructed by means of our theorem.

Corollary 6.7. *If ρ is the Rao function of a quasi-extremal curve in $\mathbb{P}^n$ having degree d and genus $g < \binom{d-n+1}{2}$ then there is a quasi-extremal curve $C \subset \mathbb{P}^n$ as in Theorem 6.5 of degree d and genus g satisfying for all integers j*

$$h^1(\mathcal{I}_C(j)) = \rho(j).$$

Proof. According to the end of the proof of Theorem 6.5 it suffices to choose D as an $(n-1)$–rope of genus $g - \binom{d-n+1}{2}$ such that

$$h^1(\mathcal{I}_D(j)) = \rho(j) \quad \text{for all}\, j < 0.$$

Comparing with Corollary 3.2 we see that such a rope D always exist by choosing the matrix A suitably. □

Of course, it is possible to construct quasi-extremal curves starting from multiple structures of degree $n-1$ on a line, which are not ropes. For example, if we consider the triple line $D \subset \mathbb{P}^4$ defined by the ideal

$$I_D = (x_0^2, x_0x_1, x_1^2, x_0x_2, x_1x_2, x_2^2 + x_1t, x_0t^a + x_1u^a)$$

we can construct a quasi-extremal curve in $\mathbb{P}^4$ just by joining it with a plane curve P of degree at least 2, subject to the condition that P and D meet in a 0-dimensional scheme of degree $\deg P$.

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