NORMAL FORM FOR SPACE CURVES IN A DOUBLE PLANE

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Abstract. This note is an attempt to relate explicitly the geometric and algebraic properties of a space curve that is contained in some double plane. We show in particular that the minimal generators of the homogeneous ideal of such a curve can be written in a very specific form. As applications we characterize the possible Hartshorne-Rao modules of curves in a double plane and the minimal curves in their even Liaison classes.

Contents

1. Introduction

Hartshorne and Schlesinger [7] have shown that a curve $C \subset \mathbb{P}^3_K$ (the projective three-space over an algebraically closed field K) contained in a double plane $2H$ determines uniquely a triple of subschemes $Z \subset C' \subset P$, all contained in the plane H, where C' and P are (planar) curves and Z is either empty or zero-dimensional, and Z is empty if and only if C is arithmetically Cohen-Macaulay (aCM). This correspondence is not bijective. The goal of this paper is to make the relationship very explicit. In fact, we show that every non-aCM curve C in a double plane is related to a certain matrix which allows to describe explicitly the homogeneous ideal of the curve C, as well as to read off the pair (Z, C') and the Hartshorne-Rao module of C. Moreover, our description of the ideal clarifies the role of P and allows to compute a graded minimal free resolution of C (including the maps). Thus, we think of our description of the ideal as a normal form for curves in a double plane. In order to explain the relations just mentioned we may assume that the plane H is defined by the ideal xR where $R = K[x, y, z, t]$ is the coordinate ring of $\mathbb{P}^3$. Thus, Z being a 2-codimensional subscheme of H, its homogeneous ideal is the ideal of the maximal minors of its Hilbert-Burch matrix $A \in S^{s,s+1}$ with entries in $S = K[y, z, t] \cong R/xR$. Denote by (x, p) the homogeneous ideal of the curve C'. Since Z is contained in C' we can write p as the determinant of a homogeneous matrix

$$p = (-1)^s \det \begin{bmatrix} & A & \\ p_1 & \dots & p_{s+1} \end{bmatrix}$$

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where $p_1, \ldots, p_{s+1} \in S$ are suitable homogeneous polynomials and the sign is used to simplify the presentation below. Moreover, the specific form of the ideal of the curve C determines a vector ${}^t(f_1, \ldots, f_{s+1})$ such that the matrix

$$B := \begin{bmatrix} & A & & f_1 \\ & & & \vdots \\ p_1 & \ldots & p_{s+1} & f_{s+1} \end{bmatrix} \in S^{s+1,s+2}$$

is homogeneous and the last column has in particular maximal degree among the columns of B. For any $u \times (u+1)$ matrix N we denote by N_i the determinant of the matrix obtained from N by deleting its i-th column. This notation allows us to summarize part of our results as follows:

Theorem 1.1.

(a) *Every non-aCM curve C in a double plane determines a homogeneous matrix*

$$B := \begin{bmatrix} & A & & f_1 \\ & & & \vdots \\ p_1 & \ldots & p_{s+1} & f_{s+1} \end{bmatrix} \in S^{s+1,s+2} \tag{1.1}$$

where $s \geq 1$, the maximal minors of $A = (a_{i,j})$ generate the ideal of Z as a subscheme of H, $B_{s+2} \neq 0$ defines $C' \subset H$, $\deg f_1 \geq \deg a_{1,1} + \sum_{j=1}^{s} \deg a_{j,j+1} - 1$ and

$$M := \begin{bmatrix} A & \begin{matrix} f_1 \\ \vdots \\ f_s \end{matrix} \end{bmatrix} \in S^{s,s+2} \tag{1.2}$$

is a presentation matrix of the Hartshorne-Rao module of C as an S-module.

(b) *Conversely, let $B \in S^{s+1,s+2}$, $s \geq 1$ be a homogeneous matrix as in (1.1) and let M be its submatrix as in 1.2. Assume:*
   (i) *the maximal minors of M do not have common zeroes in H;*
   (ii) $B_{s+2} \neq 0$;
   (iii) $\deg f_1 \geq \deg a_{1,1} + \sum_{j=1}^{s} \deg a_{j,j+1} - 1$.
   Put $p = (-1)^s B_{s+2}$ and let $h \in S$ be a non-zero homogeneous polynomial with $\deg h = \deg f_1 - (\deg a_{1,1} + \sum_{j=1}^{s} \deg a_{j,j+1} - 1)$. Then the ideal

$$(x^2, xp, phA_1 + xB_1, \ldots, phA_{s+1} + xB_{s+1})$$

defines a curve in the double plane $\{x^2 = 0\}$ associated to the triple $Z \subset C' \subset P$ where Z, C' and the Hartshorne-Rao module of C are determined as in part (a) *and P is defined by (x, ph).*

As already mentioned, writing the generators of the ideal of C as above allows to compute the minimal free resolution of C (Theorem 3.7). Moreover, it also allows to describe all possible Hartshorne-Rao modules of curves in a double plane, as indicated below.

Corollary 1.2. *A graded R-module of finite length is the Hartshorne-Rao module of a curve in the double plane $\{x^2 = 0\}$ if and only if it can be represented by a homogeneous matrix*

$$[xE_s \; M] \in R^{s,2s+2}$$

where the entries of M are in $(y,z,t)S$, satisfy the degree condition (iii) of Theorem 1.1 (b) and the maximal minors of M do not have a common zero in H (we denote by E_s the $s \times s$ identity matrix).

It is well-known that the even Liaison class of a space curve is determined by its Hartshorne-Rao module. Since the whole even Liaison class can be obtained from a minimal curve by basic double links and possibly a flat deformation, one is interested in determining the minimal curves. Below, we characterize when a minimal curve must lie in some double plane.

Proposition 1.3. *Let $N \neq 0$ be a graded R-module of finite length and denote by $\mathcal{L}_N$ the even liaison class determined by N. Then the following conditions are equivalent:*

(i) *$\mathcal{L}_N$ contains a curve lying in some double plane $2H$;*

(ii) *every minimal curve of $\mathcal{L}_N$ lies in the double plane $2H$ or, in case N is cyclic and annihilated by two independent linear forms, is an extremal curve;*

(iii) *N is one of the modules described in Corollary 1.2.*

The starting point for our results is the observation that every curve in a double plane sits in the middle of a residual sequence. We call this sequence the "expected residual sequence". In Section 3 we study and characterize the one-dimensional schemes which fit into such an expected residual sequence. For them we also determine a graded minimal free resolution. In Section 4 we analyze when such a scheme is indeed a curve, i.e. locally Cohen-Macaulay. This leads to the normal form for a curve in a double plane. Finally, we discuss the Liaison class of such a curve.

2. Standing Notation and Preliminaries

Throughout the paper we will use the following notation.

- K is an algebraically closed field.
- $R := K[x,y,z,t]$, $S := K[y,z,t] \subseteq R$. We shall identify S with R/xR when no confusion is possible.
- $C \subseteq \mathbb{P}^3 := \mathrm{Proj}(R)$ is a non-degenerate, projective curve of degree d and arithmetic genus g, where by curve we mean a pure 1-dimensional locally Cohen-Macaulay projective subscheme (i.e. without zero-dimensional components).
- H is the plane defined by the ideal xR and $2H$ denotes the closed subscheme of $\mathbb{P}^3$ whose homogeneous ideal is (x^2). It is called *double plane*.
- If $X \subseteq \mathbb{P}^3$ is a closed subscheme, then $I_X \subseteq R$ denotes the (saturated) homogeneous ideal of X and $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbb{P}^3}$ denotes the ideal sheaf of X.
- If Λ is a $u \times (u+1)$ matrix with entries in a ring, we denote by Λ_i $(i = 1, \ldots, u+1)$, the determinant of the submatrix of Λ obtained by deleting the $i-th$ column.

Let $C \subseteq 2H$ be a curve of degree d and genus g. Following [7] we associate to C a triple $Z \subset C' \subset P$ of subschemes of H, where C' and P are (planar) curves and Z is a zero-dimensional subscheme; more precisely, P is the pure 1-dimensional partof $C \cap H$, Z is the residual scheme to P in $C \cap H$, and C' is the residual scheme to the intersection of C with H. In other words we have an exact sequence:

$$0 \to \mathcal{I}_{C',\mathbb{P}^3}(-1) \to \mathcal{I}_{C,\mathbb{P}^3} \to \mathcal{I}_{C\cap H,H} \to 0, \tag{2.1}$$

where the first map is induced by multiplication by x and $\mathcal{I}_{C\cap H,H} = \mathcal{I}_{Z,H}(-P)$.

We set:

- $I_{C'} = (x, p)$, where $p \in S$, and $\delta := \deg C'$
- $I_P = (x, ph)$. Hence $d = 2\delta + \deg h$.
- Let $A = (a_{i,j})$ be a fixed Hilbert-Burch matrix of Z and let $s \geq 1$ be the number of rows of A. Note that since $Z \subseteq C'$ there are forms $p_1, \ldots, p_{s+1} \in S$ (not uniquely determined) such that $p = \sum_{i=1}^{s+1} p_i A_i$.
- We have $I_{Z,H} = (A_1, \ldots, A_{s+1})S$ and $I_{C \cap H, H} = phI_{Z,H}$.

Therefore the residual exact sequence (2.1) can be rewritten as

$$0 \to \mathcal{I}_{C',\mathbb{P}^3}(-1) \to \mathcal{I}_{C,\mathbb{P}^3} \to phI_{Z,H} \to 0 \tag{2.2}$$

and, since $H^1_*(\mathcal{I}_C) = 0$, it is equivalent to the exact sequence

$$0 \to I_{C'}(-1) \xrightarrow{x} I_C \to phI_{Z,H} \to 0 \tag{2.3}$$

Remark 2.1. Notice that $Z \neq \emptyset$ if and only if C is not aCM. Furthermore if $Z = \emptyset$, we put $I_{Z,H} = S$ and the residual sequence becomes

$$0 \to (x, p)(-1) \xrightarrow{x} I_C \to phS \to 0.$$

Then it is not difficult to see that the ideal of C has the form $I_C = (x^2, xp, ph + xf_1)$, i.e. it corresponds formally to a matrix $B = [p, f_1] \in S^{1,2}$ as in Theorem 1.1, with $s = 0$.

¿From now on we shall assume $Z \neq \emptyset$, i.e. that C is not aCM.

3. Ideals with the expected residual sequence

In this section we study ideals that behave, in some respects, as the ideal of a curve in a double plane. We describe explicitly the minimal generators for them as well as a graded minimal free resolution. Now let us fix a triple $Z \subset C' \subset P$ of subschemes of a plane H, defined by the ideal xR, and use the above notation.

Definition 3.1. Let $J \subseteq R$ be a homogeneous ideal. We say that J admits the *expected residual sequence* if it fits into an exact sequence

$$0 \to I_{C'}(-1) \xrightarrow{x} J \to phI_{Z,H} \to 0 \tag{3.1}$$

where the first map is multiplication by x. This means that $(J + xR)/xR = phI_{Z,H}$ and that $J : (x) = (x, p)$. In other words J behaves, with respect to the data p, h and Z, as the ideal of a curve in $2H$. It is clear that $J \subseteq R$ has the expected residual sequence if and only if there is an exact sequence

$$0 \to R/(x, p)(-1) \xrightarrow{x} R/J \to R/phI_Z \to 0 \tag{3.2}$$

(compare with (2.1)).

Lemma 3.2. *Let $J \subseteq R$ be a homogeneous ideal with the expected residual sequence. Then $J = I_X$, where X is a one-dimensional closed subscheme of degree d and arithmetic genus $\binom{d-\delta-1}{2} + \binom{\delta-1}{2} + \delta - \deg Z - 1$ (i.e. $p_a(X) = p_a(P) + p_a(C') + \delta - \deg Z - 1$).*

Proof. We show first that J is saturated. Let ℓ be a general element of R_1 and let $\phi \in R$ be such that $\ell\phi \in J$. Since $J + xR = phI_Z$ is saturated, it follows that $\phi = \alpha + \beta x$, where $\alpha \in J$ and $\beta \in S$. Then $\ell\beta x \in J$, whence $\ell\beta \in J : (x)$ which implies $\beta \in J : (x)$, since $J : (x)$ is saturated. Then $\beta x \in J$, whence $\phi \in J$ and J is saturated. To complete the proof it is sufficient to compute the Hilbert polynomial of X from the exact sequence (3.2). $\square$

Now we want to characterize the ideals with expected residual sequence.

Let $A \in S^{s,s+1}$ be a Hilbert-Burch matrix of Z. Since $p \in I_{Z,H}$ we can fix $p_1, \dots, p_{s+1} \in S$ such that

$$p = \begin{vmatrix} p_1 & \dots & p_{s+1} \\ & A & \end{vmatrix}.$$

For technical reasons we will write

$$p = (-1)^s \begin{vmatrix} & A & \\ p_1 & \dots & p_{s+1} \end{vmatrix}. \tag{3.3}$$

Remark 3.3. Let $J \subseteq R$ be an ideal having the expected residual sequence. Since $I_{Z,H} = (A_1, \dots, A_{s+1})S$ there are homogeneous polynomials $G_1, \dots, G_{s+1} \in S$ such that

$$J = (x^2, xp, phA_1 + xG_1, -phA_2 + xG_2, \ \dots \ , (-1)^s phA_{s+1} + xG_{s+1}) \tag{3.4}$$

and $\deg G_i = \deg(ph) + \deg A_i - 1$ for $i = 1, \dots, s+1$.

Notice that not every homogeneous ideal $J \subseteq R$ generated as above has the expected residual sequence. This depends on the choice of the G_i's, as we are going to show.

Proposition 3.4. *Consider a homogeneous ideal J as in (3.4). Then the following conditions are equivalent:*

(i) *J has the expected residual sequence;*

(ii) *$J : (x) = (x, p)$;*

(iii) *there are homogeneous polynomials $f_1, \dots, f_s \in S$ such that*

$$A \begin{bmatrix} G_1 \\ \vdots \\ G_{s+1} \end{bmatrix} = p \begin{bmatrix} f_1 \\ \vdots \\ f_s \end{bmatrix}$$

(iv) *there are homogeneous polynomials $f_1, \dots, f_{s+1} \in S$ such that the matrix*

$$B := \begin{bmatrix} & A & & f_1 \\ & & & \vdots \\ p_1 & \dots & p_{s+1} & f_{s+1} \end{bmatrix}$$

is homogeneous and $G_i = (-1)^{i-1} B_i$ for $i = 1, \dots s+1$.

Proof. (i) $\Leftrightarrow$ (ii) is clear because $J(R/xR) = phI_{Z,H}$.

(ii) $\Rightarrow$ (iv). It is sufficient to show that there are homogeneous polynomials $f_1, \dots, f_{s+1} \in S$ such that

$$\begin{bmatrix} & A & \\ p_1 & \dots & p_{s+1} \end{bmatrix} \begin{bmatrix} G_1 \\ \vdots \\ G_{s+1} \end{bmatrix} = p \begin{bmatrix} f_1 \\ \vdots \\ f_{s+1} \end{bmatrix} \tag{3.5}$$

and then to apply Cramer's rule.

Now if $(\lambda_1, \dots \lambda_{s+1})$ is a row of A we have $\sum_{i=1}^{s+1} \lambda_i (-1)^{i-1} A_i = 0$, whence

$$x \sum_{i=1}^{s+1} \lambda_i G_i \in J$$

which implies $\sum_{i=1}^{s+1} \lambda_i G_i \in (x, p)$. Since no term of this sum contains x we get $\sum_{i=1}^{s+1} \lambda_i G_i \in pS$.

It remains to prove that $\sum_{i=1}^{s+1} p_i G_i \in pS$. As above we have:

$$p^2h + x\sum_{i=1}^{s+1} p_iG_i \in J \tag{3.6}$$

whence it is sufficient to show that $p^2h \in J$.

Put $J' := J + p^2hR$. Then $J' + xR = J + xR$ by (3.6). Since $p^2h \in S$ we get $J' : (x) = (x,p)$, and hence there is an exact sequence (3.1) with J replaced by J'. It follows that the inclusion map $J \hookrightarrow J'$ is bijective, whence the conclusion.

(iv) $\Rightarrow$ (iii). By Cramer's rule we have that (i) implies (3.5), and the conclusion is obvious.

(iii) $\Rightarrow$ (ii). It is sufficient to show that if $\phi \in S$ and $x\phi \in J$ then $\phi \in (p)$.

An easy computation shows that there are $\lambda_1, \dots \lambda_{s+1} \in S$ such that

$$\phi = \sum_{i=1}^{s+1} \lambda_i G_i$$

and

$$\sum_{i=1}^{s+1} \lambda_i (-1)^{i-1} A_i = 0.$$

whence the $(\lambda_1, \dots \lambda_{s+1})$ is a linear combination of the rows of A.

By an easy computation using (iii) we get that ϕ is a multiple of p and (ii) follows. $\square$

Remark 3.5. In the proof above we have seen: If J is an ideal with the expected residual sequence, then $hp^2 \in J$.

Corollary 3.6. *J has the expected residual sequence if and only if*

$$J = (x^2, xp, phA_i + xB_i \mid i = 1, \dots, s+1)$$

where $B \in S^{s+1,s+2}$ is a matrix of the form

$$B := \begin{bmatrix} & A & & f_1 \\ & & & \vdots \\ p_1 & \dots & p_{s+1} & f_{s+1} \end{bmatrix},$$

with

$$p = (-1)^s \begin{vmatrix} & A & \\ p_1 & \dots & p_{s+1} \end{vmatrix},$$

and

- $\deg f_i = \deg a_{i,j} + \deg h + \deg A_j - 1$ *for all* $i = 1, \dots, s$ *and for all* $j = 1, \dots, s+1$,
- $\deg f_{s+1} = \deg p_j + \deg h + \deg A_j - 1 = d - \delta - 1$ *for all* $j = 1, \dots, s+1$.

Moreover $\deg f_i \geq \deg a_{i,j} + \deg A_j - 1$ *for all i and j as above.*

In particular $\deg f_i \geq \deg a_{i,j}$ *for all i, j, where equality for some pair (i,j) implies* $\deg A_j = 1$ *and* $s = 1$.

Proof. Let B be the matrix of Proposition 3.4. By item (iv) of the same Proposition we have

$$J = (x^2, xp, phA_1 + xB_1, -phA_2 - xB_2, \ \dots \ , (-1)^{s+1}phA_{s+1} + (-1)^{s+1}xB_{s+1}).$$

whence the first statement. The computation for the degrees of f_i's is an immediate consequence of Proposition 3.4(iii). The remaining statements follow by easy calculations. $\square$

Now we want to compute the minimal free resolution of a homogeneous ideal J having the expected residual sequence. We fix some notation.

Set $J = (x^2, xp,\ phA_i + xB_i \mid 1 \leq i \leq s+1)$, where B is a matrix corresponding to J (see Corollary 3.6) and let M be the matrix obtained from B by deleting the last row, i.e.

$$M := \begin{bmatrix} A & \begin{matrix} f_1 \\ \vdots \\ f_s \end{matrix} \end{bmatrix} \in S^{s,s+2}.$$

Recall that $A \in S^{s,s+1}$ is the Hilbert-Burch matrix of Z and write the minimal free resolution of $I_{Z,H}$ over S as

$$0 \to \overline{G} \xrightarrow{{}^tA} \overline{F} \to I_{Z,H} \to 0. \tag{3.7}$$

Set $F = \overline{F} \otimes_S R$, $G = \overline{G} \otimes_S R$.

Furthermore if Λ is a given matrix and $t > 0$, we denote by $I_t(\Lambda)$ the ideal generated by the minors of order t of Λ. Finally E_t denotes the $t \times t$ identity matrix.

Theorem 3.7. *Assume that J has the expected residual sequence, and let the notation be as above. Then J has the graded minimal free resolution*

$$\begin{aligned} 0 \to G(-d-1+\delta) \xrightarrow{\alpha_3} & G(-d+\delta) \oplus F(-d-1+\delta) \oplus R(-2-\delta) \xrightarrow{\alpha_2} \\ & R(-2) \oplus F(-d+\delta) \oplus R(-1-\delta) \xrightarrow{\alpha_1} J \to 0 \end{aligned} \tag{3.8}$$

where we have identified the maps with their matrices:

$$\begin{aligned} \alpha_1 \ = \ [x^2, phA_1 + xB_1,\ \ldots,\ & (-1)^{i+1}\{phA_i + xB_i\}, \ldots, \\ & (-1)^s\{phA_{s+1} + xB_{s+1}\}, -px] \in R^{s+3} \end{aligned}$$

$$\alpha_2 = \begin{bmatrix} 0 & -B_1 & \ldots & (-1)^i B_i & \ldots & (-1)^{s+1}B_{s+1} & p \\ {}^tM & & & xE_{s+1} & & & 0 \\ & A_1h & \ldots & (-1)^{i+1}A_ih & \ldots & (-1)^s A_{s+1}h & x \end{bmatrix} \in R^{s+3,2s+2}$$

$$\alpha_3 = \begin{bmatrix} -xE_s \\ {}^tM \end{bmatrix} \in R^{2s+2,s}.$$

Proof. Developing determinants along a row and using $p = (-1)^{s+2}B_{s+2}$ it is easy to check that the sequence described above is a complex. To check that it is exact we use the Buchsbaum-Eisenbud criterion (see [4], Theorem 20.9). Since $I_s(\alpha_3) \supseteq x^sR + I_s(A) = x^sR + I_{Z,H}R$, this ideal has at least codimension 3. It remains to show that $I_{s+2}(\alpha_2)$ has at least codimension 2. It is immediate to see that $x^{s+2} \in I_{s+2}(\alpha_2)$, that is $x \in \operatorname{rad}(I_{s+2}(\alpha_2))$. Moreover if N is the matrix obtained from ${}^tM$ by deleting the first row we have

$$\det \begin{bmatrix} 0 & (-1)^{s+1}B_{s+1} & p \\ & 0 & 0 \\ N & \vdots & \vdots \\ & 0 & 0 \\ & x & 0 \\ & (-1)^s A_{s+1}h & x \end{bmatrix} \in I_{s+2}(\alpha_2),$$

whence $0 \neq A_1 A_{s+1} hp \in \operatorname{rad}(I_{s+2}(\alpha_2))$. This proves that $\operatorname{codim}(I_{s+2}(\alpha_2)) \geq 2$.

Since the non-zero entries in the matrices $\alpha_1, \alpha_2, \alpha_3$ have positive degree, the exact sequence above is a minimal free resolution. □

4. Curves in the double plane

In this section we relate ideals with expected residual sequence to curves in a double plane. We use the notation as in Theorem 3.7.

Moreover for a graded R-module M we denote by $M^* := \operatorname{Hom}_R(M, R)$ the R-dual of M and by $M^\vee := \operatorname{Hom}_K(M, K)$ the graded K-dual of M.

The main result in this section is:

Theorem 4.1. *Let C be the scheme defined by an ideal J having expected residual sequence. Then we have:*

(i) $H^1_*(\mathcal{I}_C)^\vee \cong \operatorname{coker}(\alpha_3^*)(-4)$*;*

(ii) *C is a curve if and only if the maximal minors of M do not have common zeroes in H.*

(iii) *If C is a curve, then its Hartshorne-Rao module $M_C := H^1_*(\mathcal{I}_C)$ is:*

$$M_C \cong \operatorname{coker}(G^*(-\delta-2) \oplus F^*(-\delta-1) \oplus R(-d+\delta) \xrightarrow{\left[\begin{array}{cc} xE_s & M \end{array}\right]} G^*(-\delta-1)).$$

Proof. By duality $H^1_*(\mathcal{I}_C)^\vee \cong \operatorname{Ext}^2_R(J, R)(-4)$ and (i) follows by Theorem 3.7. Moreover C is a curve if and only if the graded R-module $H^1_*(\mathcal{I}_C)$ has finite length, whence (ii) follows from (i). Finally, since $M_C^\vee \cong M_C(d-2)$ (see [7], Cor. 6.2) (iii) follows from (i). □

Corollary 4.2. *M_C is minimally generated by s homogeneous elements. Moreover if $s = 1$, then M_C is a Koszul module, i.e. it has only 4 relations.*

Remark 4.3. The self-duality of M_C, proved in [7] and used in the proof of Theorem 4.1 could also be proved easily as a consequence of Theorem 3.7.

The previous Theorem and Corollary 3.6 imply part (a) of Theorem 1.1 of the introduction. Now we are going to prove part (b) of Theorem 1.1 and Corollary 1.2. As a byproduct we find an alternative proof of a key result in [7].

Lemma 4.4. *Let $A \in S^{s,s+1}$ be a homogeneous matrix. Then the following conditions are equivalent:*

(i) *there is a homogeneous matrix M of the form*

$$M = \left[\begin{array}{cc} & f_1 \\ A & \vdots \\ & f_s \end{array}\right] \in S^{s,s+2}$$

such that $I_s(M)$ is irrelevant;

(ii) *the subscheme $Z \subseteq \operatorname{Proj}(S) \cong \mathbb{P}^2$ defined by $I_s(A)$ is zero-dimensional and locally a complete intersection;*

Proof. (i) $\Rightarrow$ (ii). Since $I_s(M)$ defines a subscheme of codimension 3, by [1] Z has codimension 2, hence is zero-dimensional. Moreover, since $I_{s-1}(A) \supseteq I_s(M), Z$ is locally a complete intersection by [8], Proposition 3.2.

(ii) $\Rightarrow$ (i). By [9] (see also [8], Proposition 3.2.) we may assume that the $(s-1)\times(s-1)$ minors $A'_1, \dots, A'_{s+1}$ corresponding to the first $s-1$ rows of A generate an irrelevant ideal. Now consider the matrix

$$M := \begin{bmatrix} & 0 \\ A & \vdots \\ & 0 \\ & f_s \end{bmatrix},$$

where $f_s \in S$ is a homogeneous polynomial not vanishing at any point of Z. Then $I_s(M) \supseteq I_{Z,H} + f_s(A'_1, \dots, A'_{s+1})$, and this easily implies that $I_s(M)$ is irrelevant. $\square$

Corollary 4.5. ([7]) *Let $Z \subset C' \subset P$ be a triple as in Section 2 and let A be a Hilbert-Burch matrix of Z. Then the following conditions are equivalent:*

(i) *there is a curve $C \subseteq 2H$ corresponding to the triple;*
(ii) *Z is locally a complete intersection.*

Proof. It follows easily from Corollary 3.6, Theorem 4.1, and Lemma 4.4. $\square$

Corollary 4.6. *Let $A \in S^{s,s+1}$ and $M := \begin{bmatrix} & f_1 \\ A & \vdots \\ & f_s \end{bmatrix} \in S^{s+1,s+2}$ be homogeneous matrices such that $\deg f_1 \geq \deg a_{1,1} + \sum_{j=1}^{s} \deg a_{j,j+1} - 1$ and $I_s(M)$ is irrelevant.*

Then there is a curve in $2H$ whose Hartshorne-Rao module is presented by the matrix $[xE_s \ M]$.

Proof. By Lemma 4.4 A is the Hilbert-Burch matrix of a zero-dimensional scheme $Z \subseteq H$. Fix a homogeneous polynomial $h \in S$ such that $\deg h = \deg f_1 - (\deg a_{1,1} + \sum_{j=1}^{s} \deg a_{j,j+1} - 1)$ and construct a homogeneous matrix

$$B := \begin{bmatrix} & A & & f_1 \\ & & & \vdots \\ p_1 & \dots & p_{s+1} & f_{s+1} \end{bmatrix},$$

such that $p := (-1)^s B_{s+2}$ is non-zero. Then the homogeneous ideal

$$J = (x^2, xp, phA_i + xB_i \mid i = 1, \dots, s+1)$$

defines a curve with the required properties by Lemma 4.4 and Theorem 4.1. $\square$

The above Corollary completes the proof of Theorem 1.1 (b).

Hence Theorem 1.1 is proved. Moreover Corollary 1.2 follows by Theorem 4.1 and Corollary 4.6.

Remark 4.7. (i) Let C and B be as in Corollary 3.6. It is easy to see that elementary row operations on B produce the same curve, except adding to one of the first s rows a multiple of the last row: such an operation can change Z, hence the curve.

(ii) Similarly elementary column operations on B produce the same curve, unless $s = 1$ and $\deg f_1 = \deg a_{1,1}$ or $\deg f_1 = \deg a_{1,2}$. Indeed by the degree conditions of Corollary 3.6 it is not possible to add to one of the first $s+1$ columns a multiple of the last one, except in the particular case just mentioned. Any other column operation does not change the ideal.

We end this section by proving Proposition 1.3.

We fix a non-aCM curve $C \subseteq 2H$. This also fixes a presentation of its Hartshorne-Rao module M_C as shown in Theorem 4.1: observe that M_C has a natural structure as S-module, and as such is represented by the matrix M. First we observe:

Lemma 4.8. *We have:*

(i) $\operatorname{Ann}_S(M_C) = I_s(M)$;

(ii) $\operatorname{Ann}_R(M_C) = xR + I_s(M)R$.

Proof. Both claims follow immediately by [2]. □

Proof of Proposition 1.3.

(i) ⇔ (iii) is a consequence of Corollary 1.2 and Corollary 4.6.

(ii) ⇒ (i) follows from well-known facts on extremal curves (see e.g. [11]).

(i) ⇒ (ii). Let $C \in \mathcal{L}_N$, $C \subseteq 2H$ and let $D \in \mathcal{L}_N$ be minimal. Then D must lie on some quadric Q (see, e.g., [10], Ch. III, Prop. 3.6 and Th. 5.1). We are going to show that if C is not extremal, then Q is exactly $2H$.

Set $qR = I_Q$. Then it is easy to see that $q \in \operatorname{Ann}_R(M_D) = \operatorname{Ann}_R(M_C)$.

Assume first that Q is irreducible. Then Q must be smooth, because D is not aCM. Then D is of type $(0, b)$ (say) where $b := \deg D$. It follows that M_D is minimally generated by $b-1$ elements in degree zero (see, e.g., [5]). This implies, by Corollary 4.2, that $b-1 = s$. Now $q \in \operatorname{Ann}_R(M_C)$ is an irreducible form of degree 2, whence $s \leq 2$ by Lemma 4.8. If $s = 1$ then D is a curve of degree 2 and genus -1, hence extremal. Thus, we get $s = 2$ and the Rao function of D is easily seen to be: $\rho(0) = \rho(1) = 2$ and $\rho(j) = 0$ for $j \neq 0, 1$ (see, e.g., [5]). On the other hand by Theorem 4.1 and the degree conditions of Corollary 3.6 it follows that there is at least one j such that $\rho_C(j) \geq 3$, a contradiction.

Assume now that Q is the union of two distinct planes. Then D is extremal by [6], Example 5.7 or [7], Prop. 9.5.

Therefore Q is a double plane $2H'$. Since I_H and $I_{H'}$ are contained in $\operatorname{Ann}_R(M_D)$ by Lemma 4.8 and since D is not extremal, we have $H = H'$. □

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Cohen-Macaulay space

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