LIAISON AND RELATED TOPICS: NOTES FROM THE TORINO WORKSHOP/SCHOOL

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Contents

1. Introduction

These are the expanded and detailed notes of the lectures given by the authors during the school and workshop entitled "Liaison and Related Topics," held at the Politecnico di Torino during the period October 1-5, 2001.

The authors each gave five lectures of length 1.5 hours each. We attempted to cover liaison theory from first principles, through the main developments (especially in codimension two) and the standard applications, to the recent developments in Gorenstein liaison and a discussion of open problems. Given the extensiveness of the subject, it was not possible to go into great detail in every proof. Still, it is hoped that the material that we chose will be beneficial and illuminating for the participants, and for the reader.

1991 *Mathematics Subject Classification.* Primary 13C40, 14M06; Secondary 13C05, 13D02, 14F05, 14M05, 14M07, 14M10.

We believe that these notes will be a valuable addition to the literature, and give details and points of view that cannot be found in other expository works on this subject. Still, we would like to point out that a number of such works do exist. In particular, the interested reader should also consult [52], [72], [73], [82], [83].

We are going to describe the contents of these notes. In the expository Section 2 we discuss the origins of liaison theory, its scope and several results and problems which are more carefully treated in later sections.

Sections 3 and 4 have preparatory character. We recall several results which are used later on. In Section 3 we discuss in particular the relation between local and sheaf cohomology, and modules and sheaves. Sections 4 is devoted to Gorenstein ideals where among other things we describe various constructions of such ideals.

The discussions of liaison theory begins in Section 5. Besides giving the basic definitions we state the first results justifying the name, i.e. showing that indeed the properties of directly linked schemes can be related to each other.

Two key results of Gorenstein liaison are presented in Section 6: the somewhat surprisingly general version of basic double linkage and the fact that linearly equivalent divisors on "nice" arithmetically Cohen-Macaulay subschemes are Gorenstein linked in two steps.

The equivalence classes generated by the various concepts of linkage are discussed in Sections 7 - 10. Rao's correspondence is explained in Section 7. It is a relation between even liaison classes and certain reflexive modules/sheaves which gives necessary conditions on two subschemes for being linked in an even number of steps. In Section 8 it is shown that these conditions are also sufficient for subschemes of codimension two. It is the main open problem of Gorenstein liaison to decide if this is also true for subschemes of higher codimension. Several results are mentioned which provide evidence for an affirmative answer. Examples show that the answer is negative if one links by complete intersections only. In Section 9 we consider the structure of an even liaison class. For subschemes of codimension two it is described by the Lazarsfeld-Rao property. Moreover, we discuss the possibility of extending it to subschemes of higher codimension. In Section 10 we compare the equivalence relations generated by the different concepts of linkage. In particular, we explain how invariants for complete intersection liaison can be used to distinguish complete intersection liaison classes within one Gorenstein liaison class.

Section 11 gives a flavour of the various applications of liaison theory.

Throughout these notes we mention various open problems. Some of them and further problems related to liaison theory are stated in Section 12.

Although most of the results are true more generally for subschemes of an arithmetically Gorenstein subscheme, for simplicity we restrict ourselves to subschemes of $\mathbb{P}^n$.

Both authors were honored and delighted to be invited to give the lectures for this workshop. We are grateful to the main organizers, Gianfranco Casnati, Nadia Chiarli and Silvio Greco, for their kind hospitality. We are also grateful to the participants, especially Roberto Notari and Maria Luisa Spreafico, for their hospitality and mathematical discussions, and for their hard work in preparing this volume. Finally, we are grateful to Robin Hartshorne and Rosa Miró-Roig for helpful comments about the contents of these notes, and especially to Hartshorne for his Example 12.1.

2. Overview and History

This section will give an expository overview of the subject of liaison theory, and the subsequent sections will provide extensive detail. Liaison theory has its roots dating to more than a century ago. The greatest activity, however, has been in the last quarter century, beginning with the work of Peskine and Szpiró [91] in 1974. There are at least three perspectives on liaison that we hope to stress in these notes:

- Liaison is a very interesting subject in its own right. There are many hard open problems, and recently there is hope for a broad theory in arbitrary codimension that neatly encompasses the codimension two case, where a fairly complete picture has been understood for many years.
- Liaison is a powerful tool for constructing examples. Sometimes a hypothetical situation arises but it is not known if a concrete example exists to fit the theoretical constraints. Liaison is often used to find such an example.
- Liaison is a useful method of proof. It often happens that one can study an object by linking to something which is intrinsically easier to study. It is also a useful method of proving that an object does not exist, because if it did then a link would exist to something which can be proved to be non-existent.

Let $R = K[x_0, \dots, x_n]$ where K is a field. For a sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{P}^n}$-modules, we set

$$H^i_*(\mathcal{F}) = \bigoplus_{t \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{F}(t))$$

This is a graded R-module. One use of this module comes in the following notion.

Definition 2.1. A subscheme $X \subset \mathbb{P}^n$ is *arithmetically Cohen-Macaulay* if R/I_X is a Cohen-Macaulay ring, i.e. $\dim R/I = \operatorname{depth} R/I$, where dim is the Krull-dimension.

These notions will be discussed in greater detail in coming sections. We will see in Section 3 that X is arithmetically Cohen-Macaulay if and only if $H^i_*(\mathcal{I}_X) = 0$ for $1 \leq i \leq \dim X$. When X is arithmetically Cohen-Macaulay of codimension c, say, the minimal free resolution of I_X is as short as possible:

$$0 \to F_c \to F_{c-1} \to \cdots \to F_1 \to I_X \to 0.$$

(This follows from the Auslander-Buchsbaum theorem and the definition of a Cohen-Macaulay ring.) The *Cohen-Macaulay type* of X, or of R/I_X, is the rank of F_c. We will take as our definition that X is *arithmetically Gorenstein* if X is arithmetically Cohen-Macaulay of Cohen-Macaulay type 1, although in Section 4 we will see equivalent formulations (Proposition 4.4). For example, thanks to the Koszul resolution we know that a complete intersection is always arithmetically Gorenstein. The converse holds only in codimension two. We will discuss these notions again later, but we assume these basic ideas for the current discussion.

Liaison is, roughly, the study of unions of subschemes, and in particular what can be determined if one knows that the union is "nice." Let us begin with a very simple situation. Let C_1 and C_2 be equidimensional subschemes in $\mathbb{P}^n$ with saturated ideals $I_{C_1}, I_{C_2} \subset R$ (i.e. I_{C_1} and I_{C_2} are unmixed homogeneous ideals in R). We assume that C_1 and C_2 have no common component. We can study the union $X = C_1 \cup C_2$, with saturated ideal $I_X = I_{C_1} \cap I_{C_2}$, and the intersection $Z = C_1 \cap C_2$, defined by the ideal $I_{C_1} + I_{C_2}$. Note that this latter ideal is not necessarily saturated, so $I_Z = (I_{C_1} + I_{C_2})^{sat}$. These are related by the exact sequence

$$0 \to I_{C_1} \cap I_{C_2} \to I_{C_1} \oplus I_{C_2} \to I_{C_1} + I_{C_2} \to 0. \tag{2.1}$$

Sheafifying gives

$$0 \to \mathcal{I}_X \to \mathcal{I}_{C_1} \oplus \mathcal{I}_{C_2} \to \mathcal{I}_Z \to 0.$$

Taking cohomology and forming a direct sum over all twists, we get

$$\begin{tikzcd}
0 \arrow[r] & I_X \arrow[r] & I_{C_1} \oplus I_{C_2} \arrow[rr] \arrow[dr] & & I_Z \arrow[r] & H^1_*(\mathcal{I}_X) \arrow[r] & H^1_*(\mathcal{I}_{C_1}) \oplus H^1_*(\mathcal{I}_{C_2}) \arrow[r] & \cdots \\
& & & I_{C_1} + I_{C_2} \arrow[ur] \arrow[dr] & & & & \\
& & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

So one can see immediately that somehow $H^1_*(\mathcal{I}_X)$ (or really a submodule) measures the failure of $I_{C_1} + I_{C_2}$ to be saturated, and that if this cohomology is zero then the ideal is saturated. More observations about how submodules of $H^1_*(\mathcal{I}_X)$ measure various deficiencies can be found in [72].

Remark 2.2. We can make the following observations about our union $X = C_1 \cup C_2$:

1. If $H^1_*(\mathcal{I}_X) = 0$ (in particular if X is arithmetically Cohen-Macaulay) then $I_{C_1} + I_{C_2} = I_Z$ is saturated.
2. $I_X \subset I_{C_1}$ and $I_X \subset I_{C_2}$.
3. $[I_X : I_{C_1}] = I_{C_2}$ and $[I_X : I_{C_2}] = I_{C_1}$ since C_1 and C_2 have no common component (cf. [30] page 192).
4. It is not hard to see that we have an exact sequence
$$0 \to R/I_X \to R/I_{C_1} \oplus R/I_{C_2} \to R/(I_{C_1} + I_{C_2}) \to 0.$$
Hence we get the relations
$$\begin{aligned} \deg C_1 + \deg C_2 &= \deg X \\ p_a C_1 + p_a C_2 &= p_a X + 1 - \deg Z \quad \text{(if } C_1 \text{ and } C_2 \text{ are curves)} \end{aligned}$$
where p_a represents the arithmetic genus.
5. Even if X is arithmetically Cohen-Macaulay, it is possible that C_1 is arithmetically Cohen-Macaulay but C_2 is not arithmetically Cohen-Macaulay. For instance, consider the case where C_2 is the disjoint union of two lines in $\mathbb{P}^3$ and C_1 is a proper secant line of C_2. The union is an arithmetically Cohen-Macaulay curve of degree 3.
6. If C_1 and C_2 are allowed to have common components then observations 3 and 4 above fail. In particular, even if X is arithmetically Cohen-Macaulay, knowing something about C_1 and something about X does not allow us to say anything helpful about C_2. See Example 2.7.

The amazing fact, which is the starting point of liaison theory, is that when we restrict X further by assuming that it is arithmetically Gorenstein, then these problems can be overcome. The following definition will be re-stated in more algebraic language later (Definition 2.8).

Definition 2.3. Let C_1, C_2 be equidimensional subschemes of $\mathbb{P}^n$ having no common component. Assume that $X := C_1 \cup C_2$ is arithmetically Gorenstein. Then C_1 and C_2 are said to be *(directly) geometrically G-linked by* X, and we say that C_2 is *residual* to C_1 in X. If X is a complete intersection, we say that C_1 and C_2 are *(directly) geometrically CI-linked.*

Example 2.4. If X is the complete intersection in $\mathbb{P}^3$ of a surface consisting of the union of two planes with a surface consisting of one plane then X links a line C_1 to a different line C_2.

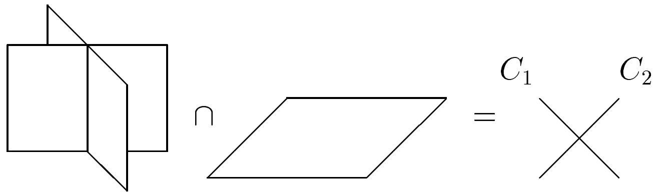


FIGURE 1. Geometric Link

Remark 2.5.
1. Given a scheme C_1, it is relatively easy (theoretically or on a computer) to find a complete intersection X containing C_1. It is much less easy to find one which gives a geometric link (see Example 2.6). In any case, X is arithmetically Cohen-Macaulay, and if one knows the degrees of the generators of I_X then one knows the degree and arithmetic genus of X and even the minimal free resolution of I_X, thanks to the Koszul resolution.
2. We will see that when X is a complete intersection, a great deal of information is passed from C_1 to C_2. For example, C_1 is arithmetically Cohen-Macaulay if and only if C_2 is arithmetically Cohen-Macaulay. We saw above that this is not true when X is merely arithmetically Cohen-Macaulay. In fact, much stronger results hold, as we shall see. An important problem in general is to find *liaison invariants*.
3. While the notion of direct links has generated a theory, *liaison theory*, that has become an active and fruitful area of study, it began as an idea that did not quite work. Originally, it was hoped that starting with *any* curve C_1 in $\mathbb{P}^3$ one could always find a way to link it to a "simpler" curve C_2 (e.g. one of smaller degree), and use information about C_2 to study C_1. Based on a suggestion of Harris, Lazarsfeld and Rao [63] showed that this idea is fatally flawed: for a general curve $C \subset \mathbb{P}^3$ of large degree, there is no simpler curve that can be obtained from C in *any* number of steps.

   However, this actually led to a structure theorem for codimension two even liaison classes [4], [68], [85], [90], often called the *Lazarsfeld-Rao property*, which is one of the main results of liaison theory.

We now return to the question of how easy it is to find a complete intersection containing a given scheme C_1 and providing a geometric link. Since our schemes are only assumed to be equidimensional, we will consider a non-reduced example.

Example 2.6. Let C_1 be a non-reduced scheme of degree two in $\mathbb{P}^3$, a so-called *double line*. It turns out (see e.g. [69], [48]) that the homogeneous ideal of C_1 is of the form

$$I_{C_1} = (x_0^2, x_0x_1, x_1^2, x_0F(x_2, x_3) - x_1G(x_2, x_3))$$

where F, G are homogeneous of the same degree, with no common factor. Suppose that $\deg F = \deg G = 100$. Then it is easy to find complete intersections I_X whose generators have degree ≤ 100; a simple example is $I_X = (x_0^2, x_1^2)$. However, any such complete intersection will have degree at least 4 along the line $x_0 = x_1 = 0$, so it cannot provide a geometric link for C_1: it is impossible to write $X = C_1 \cup C_2$ as schemes, no matter what C_2 is. However, once we look in degrees ≥ 101, geometric links are possible (since the fourth generator then enters the picture).

As this example illustrates, geometric links are too restrictive. We have to allow common components somehow. However, an algebraic observation that we made above (Remark 2.2 (3)) gives us the solution. That is, we will build our definition and theory around ideal quotients. Note first that if X is merely arithmetically Cohen-Macaulay, problems can arise, as mentioned in Remark 2.2 (6).

Example 2.7. Let $I_X = (x_0, x_1)^2 \subset K[x_0, x_1, x_2, x_3]$, let C_1 be the double line of Example 2.6 and let C_2 be the line defined by $I_{C_2} = (x_0, x_1)$. Then

$$[I_X : I_{C_1}] = I_{C_2}, \quad \text{but} \quad [I_X : I_{C_2}] = I_{C_2} \neq I_{C_1}.$$

As we will see, this sort of problem does not occur when our links are by *arithmetically Gorenstein* schemes (e.g. complete intersections). We make the following definition.

Definition 2.8. Let $C_1, C_2 \subset \mathbb{P}^n$ be subschemes with X arithmetically Gorenstein. Assume that $I_X \subset I_{C_1} \cap I_{C_2}$ and that $[I_X : I_{C_1}] = I_{C_2}$ and $[I_X : I_{C_2}] = I_{C_1}$. Then C_1 and C_2 are said to be *(directly) algebraically G-linked by* X, and we say that C_2 is *residual* to C_1 in X. We write $C_1 \overset{X}{\sim} C_2$. If X is a complete intersection, we say that C_1 and C_2 are *(directly) algebraically CI-linked.* In either case, if $C_1 = C_2$ then we say that the subscheme is *self-linked* by X.

Remark 2.9. An amazing fact, which we will prove later, is that when X is arithmetically Gorenstein (e.g. a complete intersection), then such a problem as illustrated in Example 2.7 and Remark 2.2 (5) and (6) does not arise. That is, if $I_X \subset I_{C_1}$ is arithmetically Gorenstein, and if we define $I_{C_2} := [I_X : I_{C_1}]$ then it automatically follows that $[I_X : I_{C_2}] = I_{C_1}$ whenever C_1 is equidimensional (i.e. I_{C_1} is unmixed). It also follows that $\deg C_1 + \deg C_2 = \deg X$.

One might wonder what happens if C_1 is not equidimensional. Then it turns out that

$$I_X : [I_X : I_{C_1}] = \text{top dimensional part of } C_1,$$

in other words this double ideal quotient is equal to the intersection of the primary components of I_{C_1} of minimal height (see [72] Remark 5.2.5).

Example 2.10. Let $I_X = (x_0x_1, x_0+x_1) = (x_0^2, x_0+x_1) = (x_1^2, x_0+x_1)$. Let $I_{C_1} = (x_0, x_1)$. Then $I_{C_2} := [I_X : I_{C_1}] = I_{C_1}$. That is, C_1 is self-linked by X (see Figure 2). The question of when a scheme can be self-linked is a difficult one that has been addressed by several

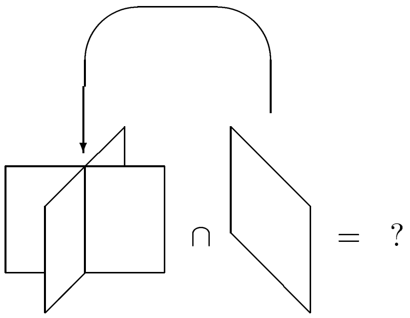


FIGURE 2. Algebraic Link

papers, e.g. [9], [27], [38], [60], [69], [96]. Most schemes are not self-linked. See also Question 4 of Section 12, and Example 12.1.

Part of Definition 2.8 is that the notion of direct linkage is symmetric. The observation above is that for most schemes it is not reflexive (i.e. most schemes are not self-linked). It is not hard to see that it is rarely transitive. Hence it is not, by itself, an equivalence relation. Liaison is the equivalence relation *generated* by direct links, i.e. the transitive closure of the direct links.

Definition 2.11. Let $C \subset \mathbb{P}^n$ be an equidimensional subscheme. The *Gorenstein liaison class of* C (or the *G-liaison class of* C) is the set of subschemes which can be obtained from C in a finite number of direct links. That is, C' is in the G-liaison class of C if there exist subschemes $C_1, \dots, C_r$ and arithmetically Gorenstein schemes $X_1, \dots, X_r, X_{r+1}$ such that

$$C \overset{X_1}{\sim} C_1 \overset{X_2}{\sim} \cdots \overset{X_r}{\sim} C_r \overset{X_{r+1}}{\sim} C'.$$

If $r+1$ is even then we say that C and C' are *evenly G-linked*, and the set of all subschemes that are evenly linked to C is the *even G-liaison class* of C. If all the links are by complete intersections then we talk about the *CI-liaison class of* C and the *even CI-liaison class of* C respectively. *Liaison* is the study of these equivalence relations.

Remark 2.12. Classically liaison was restricted to CI-links. The most complete results have been found in codimension two, especially for curves in $\mathbb{P}^3$ ([4], [68], [94], [95], [85], [90]). However, Schenzel [99] and later Nagel [85] showed that the set-up and basic results for complete intersections continue to hold for G-liaison as well, in any codimension.

As we noted earlier, in codimension two every arithmetically Gorenstein scheme is a complete intersection. Hence the complete picture which is known in codimension two belongs just as much to Gorenstein liaison theory as it does to complete intersection liaison theory!

The recent monograph [61] began the study of the important differences that arise, and led to the recent focus on G-liaison in the literature. We will describe much of this work. In particular, we will see how several results in G-liaison theory neatly generalize standard results in codimension two theory, while the corresponding statements for CI-liaison are false!

Here are some natural questions about this equivalence class, which we will discuss and answer (to the extent possible, or known) in these lectures. In the last section we will discuss several open questions. We will see that the known results very often hold for *even* liaison classes, so some of our questions focus on this case.

1. Find necessary conditions for C_1 and C_2 to be in the same (even) liaison class (i.e. find *(even) liaison invariants*). We will see that the dimension is invariant, the property of being arithmetically Cohen-Macaulay is invariant, as is the property of being locally Cohen-Macaulay, and that more generally, for an even liaison class the graded modules $H^i_*(\mathcal{I}_C)$ are essentially invariant (modulo shifts), for $1 \leq i \leq \dim C$. The situation is somewhat simpler when we assume that the schemes are locally Cohen-Macaulay. There is also a condition in terms of stable equivalence classes of certain reflexive sheaves.
2. Find sufficient conditions for C_1 and C_2 to be in the same (even) liaison class. We will see that for instance being linearly equivalent is a sufficient condition for even liaison, and that for codimension two the problem is solved. In particular, for codimension two there is a condition which is both necessary and sufficient for two schemes to be in the same even liaison class. An important question is to find a condition which is both necessary and sufficient in higher codimension, either for CI-liaison or for G-liaison. Some partial results in this direction will be discussed.
3. Is there a structure common to all even liaison classes? Again, this is known in codimension two. It is clear that the structure, as it is commonly stated in codimension two, does not hold for even G-liaison. But perhaps some weaker structure does hold.
4. Are there good applications of liaison? In codimension two we will mention a number of applications that have been given in the literature, but there are fewer known in higher codimension.
5. What are the differences and similarities between G-liaison and CI-liaison? What are the advantages and disadvantages of either one? See Remark 2.14 and Section 10.
6. Do geometric links generate the same equivalence relation as algebraic links? For CI-liaison the answer is "no" if we allow schemes that are not local complete intersections. Is the answer "yes" if we restrict to local complete intersections? And is the answer "yes" in any case for G-liaison?
7. We have seen that there are fewer nice properties when we try to allow links by arithmetically Cohen-Macaulay schemes. It is possible to define an equivalence relation using "geometric ACM links." What does this equivalence relation look like? See Remark 2.13.

Remark 2.13. We now describe the answer to Question 7 above. Clearly if we are going to study *geometric* ACM links, we have to restrict to schemes that are locally Cohen-Macaulay in addition to being equidimensional. Then we quote the following three results:

- ([106]) Any locally Cohen-Macaulay equidimensional subscheme $C \subset \mathbb{P}^n$ is ACM-linked in finitely many steps to some arithmetically Cohen-Macaulay scheme.
- ([61] Remark 2.11) Any arithmetically Cohen-Macaulay scheme is CM-linked to a complete intersection of the same dimension.

- (Classical; see [101]) Any two complete intersections of the same dimension are CI-linked in finitely many steps. (See Open Question 6 on page 64 for an interesting related question for G-liaison.)

The first of these is the deepest result. Together they show that there is only one ACM-liaison class, so there is not much to study here. Walter [106] does give a bound on the number of steps needed to pass from an arbitrary locally Cohen-Macaulay scheme to an arithmetically Cohen-Macaulay scheme, in terms of the dimension. In particular, for curves it can be done in one step!

So the most general kind of linkage for subschemes of projective space seems to be Gorenstein liaison. Recent contributions to this theory have been made by Casanellas, Hartshorne, Kleppe, Lesperance, Migliore, Miró-Roig, Nagel, Notari, Peterson, Spreafico, and others. We will describe this work in the coming sections.

Remark 2.14. To end this section, as a partial answer to Question 5, we would like to mention two results about G-liaison from [61] that are easy to state, cleanly generalize the codimension two case, and are *false* for CI-liaison.

- Let $S \subset \mathbb{P}^n$ be arithmetically Cohen-Macaulay satisfying property G_1 (so that linear equivalence is well-defined; see [50]). Let $C_1, C_2 \subset S$ be divisors such that $C_2 \in |C_1 + tH|$, where H is the class of a hyperplane section and $t \in \mathbb{Z}$. Then C_1 and C_2 are G-linked in two steps.
- Let $V \subset \mathbb{P}^n$ be a subscheme of codimension c such that I_V is the ideal of maximal minors of a $t \times (t+c-1)$ homogeneous matrix. Then V can be G-linked to a complete intersection in finitely many steps.

3. Preliminary results

The purpose of this section is to recall some concepts and results we will use later on. Among them we include a comparison of local and sheaf cohomology, geometric and algebraic hyperplane sections, local duality and k-syzygies. Furthermore, we discuss the structure of deficiency modules and introduce the notion of (cohomological) minimal shift.

Throughout we will use the following notation. A will always denote a (standard) graded K-algebra, i.e. $A = \oplus_{i \geq 0} [A]_i$ is generated (as algebra) by its elements of degree 1, $[A]_0 = K$ is a field and $[A]_i$ is the vector space of elements of degree i in A. Thus, there is a homogeneous ideal $I \subset R = K[x_0, \dots, x_n]$ such that $A \cong R/I$. The irrelevant maximal ideal of A is $\mathfrak{m} := \mathfrak{m}_A := \oplus_{i>0} [A]_i$.

If M is a graded module over the ring A it is always assumed that M is $\mathbb{Z}$-graded and A is a graded K-algebra as above. All A-modules will be finitely generated unless stated otherwise. Furthermore, it is always understood that homomorphisms between graded R-modules are morphisms in the category of graded R-modules, i.e. are graded of degree zero.

Local cohomology

There will be various instances where it is preferable to use local cohomology instead of the (possibly more familiar) sheaf cohomology. Thus we recall the definition of local cohomology and describe the comparison between both cohomologies briefly.

We start with the following

Definition 3.1. Let M be an arbitrary A module. Then we set

$$H^0_{\mathfrak{m}}(M) := \{m \in M \mid \mathfrak{m}_A^k \cdot m = 0 \text{ for some } k \in \mathbb{N}\}.$$

This construction provides the functor $H^0_{\mathfrak{m}}(\_)$ from the category of A-modules into itself. It has the following properties.

Lemma 3.2.

(a) *The functor $H^0_{\mathfrak{m}}(\_)$ is left-exact.*
(b) *$H^0_{\mathfrak{m}}(M)$ is an Artinian module.*
(c) *If M is graded then $H^0_{\mathfrak{m}}(M)$ is graded as well.*

Example 3.3. Let $I \subset R$ be an ideal with saturation $I^{sat} \subset R$ then

$$H^0_{\mathfrak{m}}(R/I) = I^{sat}/I.$$

This is left as an exercise to the reader.

Since the functor $H^0_{\mathfrak{m}}(\_)$ is left-exact one can define its right-derived functors using injective resolutions.

Definition 3.4. The i-th right derived functor of $H^0_{\mathfrak{m}}(\_)$ is called the *i-th local cohomology functor* and denoted by $H^i_{\mathfrak{m}}(\_)$.

Thus, to each short exact sequence of A-modules

$$0 \to M' \to M \to M'' \to 0$$

we have a the induced long exact cohomology sequence

$$0 \to H^0_{\mathfrak{m}}(M') \to H^0_{\mathfrak{m}}(M) \to H^0_{\mathfrak{m}}(M'') \to H^1_{\mathfrak{m}}(M') \to \dots$$

We note some further properties.

Lemma 3.5.

(a) *All $H^i_{\mathfrak{m}}(M)$ are Artinian A-modules (but often not finitely generated).*
(b) *If M is graded then all $H^i_{\mathfrak{m}}(M)$ are graded as well.*
(c) *The Krull dimension and the depth of M are cohomologically characterized by*

$$\begin{aligned} \dim M &= \max\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\} \\ \operatorname{depth} M &= \min\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\} \end{aligned}$$

Slightly more than stated in part (b) is true: The cohomology sequence associated to a short exact sequence of graded modules is an exact sequence of graded modules as well.

Part (a) implies that a local cohomology module is Noetherian if and only if it has finite length. Part (c) immediately provides the following.

Corollary 3.6. *The module M is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(M) = 0$ for all $i \neq \dim M$.*

As mentioned in the last section, a subscheme $X \subset \mathbb{P}^n$ is called *arithmetically Cohen-Macaulay* if its homogeneous coordinate ring R/I_X is Cohen-Macaulay, i.e. a Cohen-Macaulay-module over itself.

Now we want to relate local cohomology to sheaf cohomology.

The projective spectrum $X = \operatorname{Proj} A$ of a graded K-algebra A is a projective scheme of dimension $(\dim A - 1)$. Let $\mathcal{F}$ be a sheaf of modules over X. Its cohomology modules are denoted by

$$H^i_*(X, \mathcal{F}) = \bigoplus_{j \in \mathbb{Z}} H^i(X, \mathcal{F}(j)).$$

If there is no ambiguity on the scheme X we simply write $H^i_*(\mathcal{F})$.

There are two functors relating graded A-modules and sheaves of modules over X. One is the "sheafification" functor which associates to each graded A-module M the sheaf $\tilde{M}$. This functor is exact.

In the opposite direction there is the "twisted global sections" functor which associates to each sheaf $\mathcal{F}$ of modules over X the graded A-module $H^0_*(X, \mathcal{F})$. This functor is only left exact. If $\mathcal{F}$ is quasi-coherent then the sheaf $\widetilde{H^0_*(X, \mathcal{F})}$ is canonically isomorphic to $\mathcal{F}$. However, if M is a graded A-module then the module $H^0_*(X, \tilde{M})$ is not isomorphic to M in general. In fact, even if M is finitely generated, $H^0_*(X, \tilde{M})$ needs not to be finitely generated. Thus the functors $\tilde{\_\_}$ and $H^0_*(X, \_\_)$ do not establish an equivalence of categories between graded A-modules and quasi-coherent sheaves of modules over X. However, there is the following comparison result (cf. [105]).

Proposition 3.7. *Let M be a graded A-module. Then there is an exact sequence*

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to H^0_*(X, \tilde{M}) \to H^1_{\mathfrak{m}}(M) \to 0$$

and for all $i \geq 1$ there are isomorphisms

$$H^i_*(X, \tilde{M}) \cong H^{i+1}_{\mathfrak{m}}(M).$$

The result is derived from the exact sequence

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to H^0(M) \to H^1_{\mathfrak{m}}(M) \to 0$$

where $H^0(M) = \varinjlim_{n} \operatorname{Hom}_R(\mathfrak{m}^n, M)$. Note that $H^0(M) \cong H^0_*(X, \tilde{M})$.

Corollary 3.8. *Let $X \subset \mathbb{P}^n = \operatorname{Proj} R$ be a closed subscheme of dimension $d \leq n-1$. Then there are graded isomorphisms*

$$H^i_*(\mathcal{I}_X) \cong H^i_{\mathfrak{m}}(R/I_X) \quad \textit{for all } i = 1, \ldots, d+1.$$

Proof. Since $H^i_{\mathfrak{m}}(R) = 0$ if $i \leq n$ the cohomology sequence of

$$0 \to I_X \to R \to R/I_X \to 0$$

implies $H^i_{\mathfrak{m}}(R/I_X) \cong H^{i+1}_{\mathfrak{m}}(I_X)$ for all $i < n$. Thus, the last proposition yields the claim. $\square$

Remark 3.9. Let M be a graded R-module. Then its *Castelnuovo-Mumford regularity* is the number

$$\operatorname{reg} M := \min\{m \in \mathbb{Z} \mid [H^i_{\mathfrak{m}}(M)]_{j-i} = 0 \text{ for all } j > m\}.$$

For a subscheme $X \subset \mathbb{P}^n$ we put $\operatorname{reg} \mathcal{I}_X = \operatorname{reg} I_X$. The preceding corollary shows that this last definition agrees with Mumford's in [84].

It is convenient and common to use the following names.

Definition 3.10. Let $X \subset \mathbb{P}^n$ be a closed subscheme of dimension d. Then the graded R-modules $H^i_*(\mathcal{I}_X)$, $i = 1, \ldots, d$, are called the *deficiency modules* of X. If X is 1-dimensional then $H^1_*(\mathcal{I}_X)$ is also called the *Hartshorne-Rao module* of X.

The deficiency modules reflect properties of the scheme. For example, as mentioned in the first section, it follows from what we have now said (Corollary 3.6 and Corollary 3.8) that X is arithmetically Cohen-Macaulay if and only if $H^i_*(\mathcal{I}_X) = 0$ for $1 \leq i \leq \dim X$. Note that a scheme $X \subset \mathbb{P}^n$ is said to be *equidimensional* if its homogeneous ideal $I_X \subset R$ is unmixed, i.e. if all its components have the same dimension. In particular, an equidimensional scheme has no embedded components.

Lemma 3.11. *For a subscheme $X \subset \mathbb{P}^n$ we have*

(a) *X is equidimensional and locally Cohen-Macaulay if and only if all its deficiency modules have finite length.*
(b) *X is equidimensional if and only if* $\dim R/\operatorname{Ann} H^i_*(\mathcal{I}_X) \leq i - 1$ *for all* $i = 1, \ldots, \dim X$.

By a curve we always mean an equidimensional scheme of dimension 1. In particular, a curve is locally Cohen-Macaulay since by definition it does not have embedded components. Thus, we have.

Corollary 3.12. *A 1-dimensional scheme $X \subset \mathbb{P}^n$ is a curve if and only if its Hartshorne-Rao module $H^1_*(\mathcal{I}_X)$ has finite length.*

Hyperplane sections

Let $H \subset \mathbb{P}^n$ be the hyperplane defined by the linear form $l \in R$. The *geometric hyperplane section* (or simply the *hyperplane section*) of a scheme $X \subset \mathbb{P}^n$ is the subscheme $X \cap H$. We usually consider $X \cap H$ as a subscheme of $H \cong \mathbb{P}^{n-1}$, i.e. its homogeneous ideal $I_{X\cap H}$ is an ideal of $\bar{R} = R/lR$. The *algebraic hyperplane section* of X is given by the ideal $\overline{I_X} := (I_X + lR)/lR \subset \bar{R}$. $\overline{I_X}$ is not necessarily a saturated ideal. In fact, the saturation of $\overline{I_X}$ is just $I_{X\cap H}$. The difference between the hyperplane section and the algebraic hyperplane section is measured by cohomology.

Lemma 3.13.

$$H^0_{\mathfrak{m}}(R/I_X + lR) \cong I_{X\cap H}/\overline{I_X}$$

If the ground field K contains sufficiently many elements we can always find a hyperplane which is general enough with respect to a given scheme X. In particular, we get $\dim X \cap H = \dim X - 1$ if X has positive dimension. In order to relate properties of

X to the ones of its hyperplane section we note some useful facts. We use the following notation.

For a graded A-module M we denote by h_M and p_M its Hilbert function and Hilbert polynomial, respectively, where $h_M(j) = \operatorname{rank}[M]_j$. The Hilbert function and Hilbert polynomial of a subscheme $X \subset \mathbb{P}^n$ are the corresponding functions of its homogeneous coordinate ring R/I_X. For a numerical function $h : \mathbb{Z} \to \mathbb{Z}$ we define its first difference by $\Delta h(j) = h(j) - h(j-1)$ and the higher differences by $\Delta^i h = \Delta(\Delta^{i-1} h)$ and $\Delta^0 h = h$.

Remark 3.14. Suppose K is an infinite field and let $H \subset \mathbb{P}^n$ be a hyperplane.

(i) If $\dim X > 0$ and H is general enough then we have

$$I_{X \cap H} = \overline{I_X} \quad \text{if and only if} \quad H^1_*(\mathcal{I}_X) = 0.$$

(ii) If $X \subset \mathbb{P}^n$ is locally or arithmetically Cohen-Macaulay of positive dimension then $X \cap H$ has the same property for a general hyperplane H. The converse is false in general.

(iii) Suppose $X \subset \mathbb{P}^n$ is arithmetically Cohen-Macaulay of dimension d. Let $l_1, \ldots, l_{d+1} \in R$ be linear forms such that $\bar{A} = R/(I_X + (l_1, \ldots, l_{d+1}))$ has dimension zero. Then $\bar{A}$ is called an *Artinian reduction* of R/I_X. For its Hilbert function we have $h_{\bar{A}} = \Delta^{d+1} h_{R/I_X}$.

Minimal free resolutions

Let $R = K[x_0, \ldots, x_n]$ be the polynomial ring. By our standard conventions a homomorphism $\varphi : M \to N$ of graded R-modules is graded of degree zero, i.e. $\varphi([M]_j) \subset [N]_j$ for all integers j. Thus, we have to use degree shifts when we consider the homomorphism $R(-i) \to R$ given by multiplication by x_0^i. Observe that $R(-i)$ is not a graded K-algebra unless $i = 0$.

Definition 3.15. Let M be a graded R-module. Then $N \neq 0$ is said to be a *k-syzygy of M* (as R-module) if there is an exact sequence of graded R-modules

$$0 \to N \to F_k \xrightarrow{\varphi_k} F_{k-1} \to \ldots \to F_1 \xrightarrow{\varphi_1} M \to 0$$

where the modules $F_i, i = 1, \ldots, k$, are free R-modules. A module is called a *k-syzygy* if it is a k-syzygy of some module.

Note that a $(k+1)$-syzygy is also a k-syzygy (not for the same module). Moreover, every k-syzygy N is a maximal R-module, i.e. $\dim N = \dim R$.

Chopping long exact sequences into short ones we easily obtain

Lemma 3.16. *If N is a k-syzygy of the R-module M then*

$$H^i_{\mathfrak{m}}(N) \cong H^{i-k}_{\mathfrak{m}}(M) \quad \text{for all } i < \dim R.$$

If follows that the depth of a k-syzygy is at least k.

The next concept ensures uniqueness properties.

Definition 3.17. Let $\varphi : F \to M$ be a homomorphism of R-modules where F is free. Then φ is said to be a *minimal* homomorphism if $\varphi \otimes id_{R/\mathfrak{m}} : F/\mathfrak{m}F \to M/\mathfrak{m}M$ is the zero map in case M is free and an isomorphism in case φ is surjective.

In the situation of the definition above, N is said to be a *minimal k-syzygy* of M if the morphisms $\varphi_i, i = 1, \ldots, k$, are minimal. If N happens to be free then the exact sequence is called a *minimal free resolution* of M.

Nakayama's lemma implies easily that minimal k-syzygies of M are unique up to isomorphism and that a minimal free resolution is unique up to isomorphism of complexes.

Note that every finitely generated projective R-module is free.

Remark 3.18. Let

$$0 \to F_s \xrightarrow{\varphi_s} F_{s-1} \to \ldots \to F_1 \xrightarrow{\varphi_1} F_0 \to M \to 0$$

be a free resolution of M. Then it is minimal if and only if (after choosing bases for $F_0, \ldots, F_s$) the matrices representing $\varphi_1, \ldots, \varphi_s$ have entries in the maximal ideal $\mathfrak{m} = (x_0, \ldots, x_n)$ only.

Duality results

Later on we will often use some duality results. Here we state them only for the polynomial ring $R = K[x_0, \ldots, x_n]$. However, they are true, suitably adapted, over any graded Gorenstein K-algebra.

Let M be a graded R-module. Then we will consider two types of dual modules, the R-dual $M^* := \operatorname{Hom}_R(M, R)$ and the K-dual $M^\vee := \oplus_{j \in \mathbb{Z}} \operatorname{Hom}_K([M]_{-j}, K)$.

Now we can state a version of Serre duality (cf. [100], [105]).

Proposition 3.19. *Let M be a graded R-module. Then for all $i \in \mathbb{Z}$, we have natural isomorphisms of graded R-modules*

$$H^i_{\mathfrak{m}}(M)^\vee \cong \operatorname{Ext}^{n+1-i}_R(M, R)(-n-1).$$

The K-dual of the top cohomology module plays a particular role.

Definition 3.20. The module $K_M := \operatorname{Ext}^{n+1-\dim M}_R(M, R)(-n-1)$ is called the *canonical module* of M. The canonical module K_X of a subscheme $X \subset \mathbb{P}^n$ is defined as K_{R/I_X}.

Remark 3.21. (i) For a subscheme $X \subset \mathbb{P}^n$ the sheaf $\omega_X := \widetilde{K_X}$ is the dualizing sheaf of X.

(ii) If $X \subset \mathbb{P}^n$ is arithmetically Cohen-Macaulay with minimal free resolution

$$0 \to F_c \xrightarrow{\varphi_c} F_{c-1} \to \ldots \to F_1 \to I_X \to 0$$

then dualizing with respect to R provides the complex

$$0 \to R \to F_1^* \to \ldots \to F_{c-1}^* \xrightarrow{\varphi_c^*} F_c^* \to \operatorname{coker} \varphi_c^* \to 0$$

which is a minimal free resolution of $\operatorname{coker} \varphi_c^* \cong K_X(n+1)$.

If the scheme X is equidimensional and locally Cohen-Macaulay, one can relate the cohomology modules of X and its canonical module. More generally, we have ([100], Korollar 3.1.3).

Proposition 3.22. *Let M be a graded R-module such that $H_{\mathfrak{m}}^i(M)$ has finite length if $i \neq d = \dim M$. Then there are canonical isomorphisms for $i = 2, \dots, d-1$*

$$H_{\mathfrak{m}}^{d+1-i}(K_M) \cong H_{\mathfrak{m}}^i(M)^\vee.$$

Observe that the first cohomology $H_{\mathfrak{m}}^1(M)$ is not involved in the statement above.

Restrictions for deficiency modules

Roughly speaking, it will turn out that there are no restrictions on the module structure of deficiency modules, but there are restrictions on the degrees where non-vanishing pieces can occur.

In the following result we will assume $c \leq n-1$ because subschemes of $\mathbb{P}^n$ with codimension n are arithmetically Cohen-Macaulay.

Proposition 3.23. *Suppose the ground field K is infinite. Let c be an integer with $2 \leq c \leq n-1$ and let $M_1, \dots, M_{n-c}$ be graded R-modules of finite length. Then there is an integral locally Cohen-Macaulay subscheme $X \subset \mathbb{P}^n$ of codimension c such that*

$$H_*^i(\mathcal{I}_X) \cong M_i(-t) \quad \textit{for all } i = 1, \dots, n-c$$

for some integer t.

Proof. Choose a smooth complete intersection $V \subset \mathbb{P}^n$ such that

$$I_V = (f_1, \dots, f_{c-2}) \subset \bigcap_{i=1}^{n-c} \operatorname{Ann} M_i$$

where $I_V = 0$ if $c = 2$.

Let N_i denote a $(i+1)$-syzygy of M_i as R/I_V-module and let r be the rank of $N = \oplus_{i=1}^{n-c} N_i$. For $s \ll 0$ the cokernel of a general map $\varphi : R^{r-1} \to N$ is torsion-free of rank one, i.e. isomorphic to $I(t)$ for some integer t where $I \subset A = R/I_V$ is an ideal such that $\dim A/I = \dim A - 2$. Moreover, I is a prime ideal by Bertini's theorem. The preimage of I under the canonical epimorphism $R \to A$ is the defining ideal of a subscheme $X \subset \mathbb{P}^n$ having the required properties. For details we refer to [79]. $\square$

Remark 3.24. (i) The previous result can be generalized as follows. Let $M_1, \dots, M_{n-c}$ be graded (not necessarily finitely generated) R-modules such that $M_i^\vee$ is finitely generated of dimension $\leq i-1$ for all $i = 1, \dots, n-c$. Then there is an equidimensional subscheme $X \subset \mathbb{P}^n$ of codimension c such that

$$H_*^i(\mathcal{I}_X) \cong M_i(-t) \quad \text{for all } i = 1, \dots, n-c$$

for some integer t. Details will appear in [86]. Note that the condition on the modules $M_1, \dots, M_{n-c}$ is necessary according to Lemma 3.11.

(ii) A more general version of Proposition 3.23 for subschemes of codimension two is shown in [36].

Now we want to consider the question of which numbers t can occur in Proposition 3.23. The next result implies that with t also $t+1$ occurs. The name of the statement will be explained later on.

Lemma 3.25 (Basic double link)**.** *Let $0 \neq J \subset I \subset R$ be homogeneous ideals such that* $\operatorname{codim} I = \operatorname{codim} J + 1$ *and R/J is Cohen-Macaulay. Let $f \in R$ be a homogeneous element of degree d such that $J : f = J$. Then the ideal $\tilde{I} := J + fI$ satisfies* $\operatorname{codim} \tilde{I} = \operatorname{codim} I$ *and*

$$H^i_{\mathfrak{m}}(R/\tilde{I}) \cong H^i_{\mathfrak{m}}(R/I)(-d) \quad \textit{for all } i < \dim R/I.$$

In particular, I is unmixed if and only if $\tilde{I}$ is unmixed.

Proof. Consider the sequence

$$0 \to J(-d) \xrightarrow{\varphi} J \oplus I(-d) \xrightarrow{\psi} \tilde{I} \to 0 \tag{3.1}$$

where φ and ψ are defined by $\varphi(j) = (fj, j)$ and $\psi(j,i) = j - fi$. It is easy to check that this sequence is exact. Its cohomology sequence implies the claim on the dimension and cohomology of $R/\tilde{I}$. The last claim follows by Lemma 3.11. □

Proposition 3.26. *Suppose that K is infinite. Let $M_\bullet = (M_1, \dots, M_{n-c})$ $(2 \leq c < n)$ be a vector of graded (not necessarily finitely generated) R-modules such that $M_i^\vee$ is finitely generated of dimension $\leq i-1$ for all $i = 1, \dots, n-c$ and not all of these modules are trivial. Then there is an integer t_0 such that there is an equidimensional subscheme $X \subset \mathbb{P}^n$ of codimension c with*

$$H^i_*(\mathcal{I}_X) \cong M_i(-t) \quad \textit{for all } i = 1, \dots, n-c$$

for some integer t if and only if $t \geq t_0$.

Proof. If the ground field K is infinite we can choose the element f in Lemma 3.25 as a linear form. Thus, in spite of this lemma and Remark 3.24 it suffices to show that

$$H^i_*(\mathcal{I}_X) \cong M_i(-t) \quad \text{for all } i = 1, \dots, n-c$$

is impossible for a subscheme $X \subset \mathbb{P}^n$ of codimension c if $t \ll 0$. But this follows if $\dim X = 1$ because we have for every curve $C \subset \mathbb{P}^n$

$$h^1(\mathcal{I}_C(j-1)) \leq \max\{0, h^1(\mathcal{I}_C(j)) - 1\} \quad \text{if } j \leq 0 \tag{3.2}$$

by [21], Lemma 3.4 or [70]. By taking general hyperplane sections of X, the general case is easily reduced to the case of curves. See also Proposition 1.4 of [18]. □

The last result allows us to make the following definition.

Definition 3.27. The integer t_0, which by Proposition 3.26 is uniquely determined, is called the *(cohomological) minimal shift* of $M_\bullet$.

Example 3.28. Let $M_\bullet = (K)$. Then the estimate (3.2) for the first cohomology of a curve in the last proof shows that the minimal shift t_0 of $M_\bullet$ must be non-negative. Since we have for a pair C of skew lines $H^1_*(\mathcal{I}_C) \cong K$ we obtain $t_0 = 0$ as minimal shift of (K).

4. Gorenstein Ideals

Before we can begin the discussion of Gorenstein liaison, we will need some basic facts about Gorenstein ideals and Gorenstein algebras. In this section we will give the definitions, properties, constructions, examples and applications which will be used or discussed in the coming sections. Most of the material discussed here is treated in more detail in [72].

We saw in Remark 3.14 that if X is arithmetically Cohen-Macaulay of dimension d with coordinate ring $A = R/I_X$ then we have the *Artinian reduction* $\bar{A}$ of X (or of R/I_X). Its Hilbert function was given as $h_{\bar{A}} = \Delta^{d+1} h_{R/I_X}$. Since $\bar{A}$ is finite dimensional as a K-vector space, we have that $h_{\bar{A}}$ is a finite sequence of integers

$$1 \;\; c \;\; h_2 \;\; h_3 \;\; \dots \;\; h_s \;\; 0 \;\; \dots$$

This sequence is called the *h-vector* of X, or of A. In particular, c is the *embedding codimension* of X. In other words, c is the codimension of X inside the smallest linear space containing it. Of course, the Hilbert function of X can be recovered from the h-vector by "integrating."

Now suppose that X is arithmetically Cohen-Macaulay and non-degenerate in $\mathbb{P}^n$, of codimension c, and that R/I_X has minimal free resolution

$$0 \to F_c \to F_{c-1} \to \dots \to F_1 \to R \to R/I_X \to 0.$$

Suppose that $F_c = \bigoplus_{i=1}^r R(-a_i)$ and let $a = \max_i \{a_i\}$. As mentioned in Section 2, $r = \operatorname{rank} F_c$ is called the *Cohen-Macaulay type* of X (or of A). Furthermore, we have the relation

$$a - c = s = \operatorname{reg} \mathcal{I}_X - 1 \tag{4.1}$$

where s is the last degree in which the h-vector is non-zero and $\operatorname{reg} \mathcal{I}_X$ is the Castelnuovo-Mumford regularity of $\mathcal{I}_X$ (cf. Remark 3.9). We now formally make the definition referred to in Section 2:

Definition 4.1. The subscheme $X \subset \mathbb{P}^n$ is *arithmetically Gorenstein* if it is arithmetically Cohen-Macaulay of Cohen-Macaulay type 1. We often say that I_X is Gorenstein or I_X is arithmetically Gorenstein.

Example 4.2. A line in $\mathbb{P}^3$ is arithmetically Gorenstein since its minimal free resolution is

$$0 \to R(-2) \to R(-1)^2 \to I_X \to 0,$$

and $R(-2)$ has rank 1. More generally, any complete intersection in $\mathbb{P}^n$ is arithmetically Gorenstein thanks to the Koszul resolution. The last free module in the resolution of the complete intersection of forms of degree $d_1, \dots, d_c$ is $R(-d_1 - \dots - d_c)$.

Remark 4.3. In Remark 3.14 (ii) it was noted that if X is arithmetically Cohen-Macaulay of dimension ≥ 1 then the general hyperplane section $X \cap H$ is also arithmetically Cohen-Macaulay. (In fact this is true for any proper hyperplane section.) It was remarked that the converse is false in general. However, there are situations in which the converse does hold.

First, if X is assumed to be equidimensional (i.e. I_X is unmixed) and locally Cohen-Macaulay of dimension ≥ 2 then it is not hard to show that the converse holds. Indeed, let L be a general linear form defining the hyperplane H and consider the exact sequence

$$H^i(\mathcal{I}_X(s-1)) \xrightarrow{\times L} H^i(\mathcal{I}_X(s)) \to H^i(\mathcal{I}_{X\cap H|H}(s)) \to H^{i+1}(\mathcal{I}_X(s-1)) \xrightarrow{\times L} H^{i+1}(\mathcal{I}_X(s))$$

If $X \cap H$ is arithmetically Cohen-Macaulay and $1 \leq i \leq \dim X \cap H = \dim X - 1$ then the multiplication map on the left is surjective for all s and the one on the right is injective for all s. Both of these are impossible unless X is itself arithmetically Cohen-Macaulay, because $H^i(\mathcal{I}_X)$ has finite length for $1 \leq i \leq \dim X$ (cf. Lemma 3.11).

Obviously if X is the union of an arithmetically Cohen-Macaulay scheme and a point (possibly embedded) then it is not arithmetically Cohen-Macaulay but its general hyperplane section is arithmetically Cohen-Macaulay. Also, clearly if X is a curve which is not arithmetically Cohen-Macaulay then its general hyperplane section is arithmetically Cohen-Macaulay since it is a finite set of points, but again, X is not arithmetically Cohen-Macaulay. A fascinating question is whether there are conditions on $X \cap H$ which *force* a curve X to be arithmetically Cohen-Macaulay. The best results in this direction come when $X \cap H$ is arithmetically Gorenstein. Several authors have contributed to this question, but we mention in particular [58] and [103].

There are several other conditions which are equivalent to being arithmetically Cohen-Macaulay with Cohen-Macaulay type 1, and which could be used in the definition of arithmetically Gorenstein subschemes of $\mathbb{P}^n$.

Proposition 4.4. *Let $X \subset \mathbb{P}^n$ be arithmetically Cohen-Macaulay. The following are equivalent:*

(i) *X has Cohen-Macaulay type 1 (i.e. is arithmetically Gorenstein);*
(ii) *$R/I_X \cong K_X(\ell)$ for some $\ell \in \mathbb{Z}$, where K_X is the canonical module of X (cf. Definition 3.20);*
(iii) *The minimal free resolution of R/I_X is self-dual up to twisting by $n+1$.*

Proof. Note that ℓ is whatever twist moves the module so that it starts in degree 0. The main facts used in the proof are that

$$\begin{array}{rrcl} & K_X & = & \operatorname{Ext}^c_R(R/I_X, R)(-n-1) \\ \text{and} & I_X & = & \operatorname{Ann}_R(K_X) \end{array}$$

Details of the proof can be found in [72]. ☐

Corollary 4.5. *Let X be arithmetically Gorenstein. Then $\mathcal{O}_X \cong \omega_X(\ell)$ for some $\ell \in \mathcal{Z}$.*

Corollary 4.6. *Let X be arithmetically Gorenstein. Then the h-vector of X is symmetric.*

Proof. This follows from the fact that the Gorenstein property is preserved in passing to the Artinian reduction, and the Hilbert function of the canonical module of the Artinian reduction is given by reading the h-vector backwards (cf. [72]). ☐

The integer ℓ in Proposition 4.4 is related to the integers in the equation (4.1). In fact, we have

Corollary 4.7. *Let X be arithmetically Gorenstein with minimal free resolution*

$$0 \to R(-a) \to F_{c-1} \to \cdots \to F_1 \to R \to R/I_X \to 0$$

and assume that $\mathcal{O}_X \cong \omega_X(\ell)$. Then $\ell = n+1-a$.

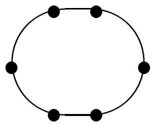
If A is Gorenstein then the integer s, the last degree in which the h-vector is non-zero, is called the *socle degree* of $\bar{A}$, the Artinian reduction of $A = R/I_X$.

There is a very useful criterion for zeroschemes to be arithmetically Gorenstein. To explain it, we will need a new notion. For now we will assume that our zeroschemes are reduced, although the necessity for this was removed by Kreuzer [62].

Definition 4.8. Let $Z \subset \mathbb{P}^n$ be a finite reduced set of points. Assume that $s+1 = \operatorname{reg}(\mathcal{I}_X)$, i.e. s is the last degree in which the h-vector of Z is non-zero. Then Z has the *Cayley-Bacharach property* (CB) if, for every subset $Y \subset Z$ consisting of $|Z|-1$ points, we have $h_{R/I_Y}(s-1) = h_{R/I_Z}(s-1)$. Z has the *Uniform Position property* (UPP) if any two subsets Y, Y' of (the same) arbitrary cardinality have the same Hilbert function, which necessarily is

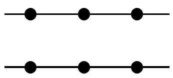
$$h_{R/I_Y}(t) = \min\{h_{R/I_Z}(t), |Y|\} \quad \text{for all } t.$$

Example 4.9. The Cayley-Bacharach property is a weaker version of the Uniform Position Property. For example, in $\mathbb{P}^2$ consider the following examples.



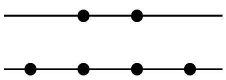
h-vector 1 2 2 1 (complete intersection on a conic)

This has UPP.



h-vector 1 2 2 1 (complete intersection)

This has CB but not UPP.



h-vector 1 2 2 1

This has neither CB nor UPP.

Theorem 4.10 ([31])**.** *A reduced set of points Z is arithmetically Gorenstein if and only if its h-vector is symmetric and it has the Cayley-Bacharach property.*

Example 4.11. A set of $n+2$ points in $\mathbb{P}^n$ in linear general position is arithmetically Gorenstein. In particular, a set of 5 points in $\mathbb{P}^3$ is arithmetically Gorenstein, so we see that 4 points in linear general position are G-linked to one point. This was the first illustration of the fact that G-liaison behaves quite differently from CI-liaison, since it follows from work of Ulrich and others that 4 points in linear general position are not CI-linked to a single point in any number of steps.

Remark 4.12. Theorem 4.10 was used by Bocci and Dalzotto [12] to produce (and verify) nice concrete examples of arithmetically Gorenstein sets of points in $\mathbb{P}^3$, and this work is described in this volume. Generalizations of this construction have been given by Bocci, Dalzotto, Notari and Spreafico [13].

A very useful construction of arithmetically Gorenstein schemes is the following.

Theorem 4.13. (Sums of geometrically linked ideals) *Let $V_1, V_2 \subset \mathbb{P}^n$ be arithmetically Cohen-Macaulay subschemes of codimension c with no common component. Assume that $V_1 \cup V_2 = X$ is arithmetically Gorenstein, i.e. $I_{V_1} \cap I_{V_2} = I_X$ with R/I_X Gorenstein. Then $I_{V_1} + I_{V_2}$ is Gorenstein of codimension $c+1$ (i.e. $V_1 \cap V_2$ is arithmetically Gorenstein).*

Proof. From the exact sequence (2.1) we can build up the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & R(-a) \arrow[d] & A_c \oplus B_c \arrow[d] & & \\
 & F_{c-1} \arrow[d] & A_{c-1} \oplus B_{c-1} \arrow[d] & & \\
 & \vdots \arrow[d] & \vdots \arrow[d] & & \\
 & F_1 \arrow[d] & A_1 \oplus B_1 \arrow[d] & & \\
0 \arrow[r] & I_X \arrow[r] \arrow[d] & I_{V_1} \oplus I_{V_2} \arrow[r] \arrow[d] & I_{V_1} + I_{V_2} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The mapping cone then gives the long exact sequence

$$0 \to R(-a) \to F_{c-1} \oplus A_c \oplus B_c \to F_{c-2} \oplus A_{c-1} \oplus B_{c-1} \to \dots$$

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & F_1 \oplus A_2 \oplus B_2 \arrow[r] & A_1 \oplus B_1 \arrow[rr] \arrow[dr] & & R \arrow[r] & R/(I_{V_1} + I_{V_2}) \arrow[r] & 0 \\
 & & & I_{V_1} + I_{V_2} \arrow[ur] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Of course there may be some splitting. However, $V_1 \cap V_2$ has codimension $\geq c+1$ since V_1 and V_2 have no common component. This resolution has homological dimension at most $c+1$. Therefore it has homological dimension exactly $c+1$ and $V_1 \cap V_2$ is arithmetically Cohen-Macaulay of codimension $c+1$ with Cohen-Macaulay type 1, i.e. is arithmetically Gorenstein. □

This construction has been used to good effect in constructing arithmetically Gorenstein schemes with nice properties. To illustrate, let us consider some natural questions.

Question 4.14. What are the possible Hilbert functions (resp. minimal free resolutions) of Artinian Gorenstein ideals?

Question 4.15. What are the possible Hilbert functions (resp. minimal free resolutions) of the ideals of *reduced* arithmetically Gorenstein subschemes of $\mathbb{P}^n$?

The general question of which Artinian ideals, or which properties of Artinian ideals, can be lifted to reduced sets of points is a very interesting one. We will discuss some of the known answers to Questions 4.14 and 4.15 according to the codimension.

Case I: Codimension Two

What are the possible arithmetically Gorenstein subschemes X? We know the beginning and the end of the resolution:

$$0 \to R(-a) \to (\ ??\ ) \to R \to R/I_X \to 0.$$

By considering the rank, the middle term in this resolution has to have rank 2. Therefore, we have established the well known fact (mentioned before) that *every arithmetically Gorenstein subscheme of* $\mathbb{P}^n$ *of codimension two is a complete intersection.* This answers the question about the minimal free resolution, so the Hilbert functions are known as well. In fact, the h-vectors must be symmetric of the form

$$1\ \ 2\ \ 3\ \ \ldots s-1\ \ s\ \ s\ \ \ldots\ \ s\ \ s-1\ \ \ldots\ \ 3\ \ 2\ \ 1.$$

Case II: Codimension Three

Everything that is known in this case follows from the famous structure theorem of Buchsbaum and Eisenbud [22]. For a Gorenstein ideal I we have a minimal free resolution

$$0 \to R(-a) \to F_2 \xrightarrow{A} F_1 \to R \to R/I \to 0.$$

One can choose bases so that A is skew-symmetric. In particular, the number of generators must be odd! Diesel used this result to completely describe the possible graded Betti numbers for Artinian Gorenstein ideals. De Negri and Valla (and others) described the possible Hilbert functions. In particular, not only must it be symmetric, but the "first half" must be a so-called *differentiable O-sequence.* This means that the first difference of the "first half" of the Hilbert function must grow in a way that is permissible for standard K-algebras. For example, the sequence

$$1\ \ 3\ \ 6\ \ 7\ \ 9\ \ 7\ \ 6\ \ 3\ \ 1$$

is not a possible Hilbert function for an Artinian Gorenstein algebra (even though it itself satisfies Macaulay's growth condition) since the first difference of the "first half" is

$$1\ \ 2\ \ 3\ \ 1\ \ 2$$

and the growth from degree 3 to degree 4 in the first difference exceeds Macaulay's growth condition (cf. [66]). This describes the answers to Question 4.14.

For Question 4.15, Geramita and Migliore [44] showed that any set of graded Betti numbers which occurs at the Artinian level in fact occurs for a reduced set of points (or for a stick figure curve, or more generally a "generalized stick figure" configuration of linear varieties). The idea was to use Theorem 4.13 and add the ideals of geometrically linked stick figure curves in $\mathbb{P}^3$ (or suitable surfaces in $\mathbb{P}^4$, etc.) in suitable constructed complete intersections. Ragusa and Zappalà [92], [93] have used the "sum of geometrically linked ideals" construction to obtain other nice results on the Hilbert functions and resolutions of codimension three Gorenstein ideals.

Case III: Codimension ≥ 4

To date no one has determined what Hilbert functions can occur, so certainly we do not know what minimal free resolutions can occur. In codimension ≥ 5 it is known that the Hilbert function of an Artinian Gorenstein algebra does not even have to be unimodal [10], [14], [15]. This is open in codimension 4. However, the situation that one would expect to be the "general" one is better understood:

Definition 4.16. An Artinian algebra R/I has the *Weak Lefschetz property* if, for a general linear form L, the multiplication map

$$\times L : (R/I)_i \to (R/I)_{i+1}$$

has maximal rank, for all i.

When the socle degree is fixed, a result of Watanabe [108] says that the "general" Artinian Gorenstein algebra has the Weak Lefschetz property.

When the whole Hilbert function is fixed, a similar result is not possible in general because the parameter space for the corresponding Gorenstein algebras can have several components if the codimension is at least four. However, since having the Weak Lefschetz property is an open condition by semicontinuity, the general Artinian Gorenstein algebra of a component has the Weak Lefschetz property if and only if the component contains one algebra with this property.

In any case, Harima [46] classified the possible Hilbert functions for Artinian Gorenstein algebras with the Weak Lefschetz property, in any codimension. In particular, he showed that these Hilbert functions are precisely the Stanley-Iarrobino (SI) sequences, namely they are symmetric, unimodal and the "first half" is a differentiable O-sequence.

For Question 4.15, Migliore and Nagel [77] have shown that any SI-sequence is the h-vector of some arithmetically Gorenstein reduced set of points, or more generally a reduced union of linear varieties. The method of proof again used sums of geometrically linked ideals, but the new twist here was that the ideals were G-linked and not CI-linked. Furthermore, they gave sharp bounds on the graded Betti numbers of Gorenstein ideals of any codimension, *among ideals with the Weak Lefschetz property.* Partial results along these lines had been obtained by Geramita, Harima and Shin [40]. In codimension 4, Iarrobino and Srinivasan (in progress) have some results on the possible resolutions. There remains a great deal to do in this area.

Remark 4.17. Theorem 4.13 shows how to use geometrically linked, codimension c, arithmetically Cohen-Macaulay subschemes of $\mathbb{P}^n$ to construct a codimension $c+1$ arithmetically Gorenstein subscheme. Later, in Corollary 5.19, we will see how to use very special linked arithmetically Cohen-Macaulay codimension c subschemes (not necessarily geometrically linked) to construct an arithmetically Gorenstein subscheme which is also of codimension c. In fact, every Gorenstein ideal arises in this way (Remark 5.20)!

One problem with the construction of Theorem 4.13 is that it is very desirable, from the point of view of liaison, to be able to start with a scheme V and find a "good" (which often means "small") Gorenstein scheme X containing it. This is not so easy to do with sums of geometrically linked ideals. Another very useful construction for Gorenstein ideals potentially will solve this problem (based on experimental evidence). To describe it we will need a little preparation.

Consider a homogeneous map

$$\bigoplus_{i=1}^{t+r} R(-a_i) \xrightarrow{\phi} \bigoplus_{j=1}^{t} R(-b_j).$$

The map ϕ is represented by a homogeneous $t \times (t+r)$ matrix. We assume furthermore that the ideal of maximal minors of ϕ defines a scheme of the "expected" codimension $r+1$. Let B_ϕ be the kernel of ϕ. Then B_ϕ is a *Buchsbaum-Rim module*. Let M_ϕ be the cokernel of ϕ. We have an exact sequence

$$\begin{array}{ccccccccc} 0 & \to & B_\phi & \to & \bigoplus_{i=1}^{t+r} R(-a_i) & \xrightarrow{\phi} & \bigoplus_{j=1}^{t} R(-b_j) & \to & M_\phi & \to & 0. \\ & & & & \| & & \| & & & & \\ & & & & F & & G & & & & \end{array}$$

Sheafifying this gives

$$0 \to \tilde{B}_\phi \to \bigoplus_{i=1}^{t+r} \mathcal{O}_{\mathbb{P}^n}(-a_i) \xrightarrow{\phi} \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}^n}(-b_j) \to \tilde{M}_\phi \to 0.$$

If $r = n$ then $\tilde{M} = 0$ and $\tilde{B}_\phi$ is locally free. In any case, $\tilde{B}_\phi$ is the *Buchsbaum-Rim sheaf associated to ϕ.*

Theorem 4.18 ([78]). **(Sections of Buchsbaum-Rim sheaves)** *Assume that r is odd. Let s be a* regular *section of $\tilde{B}_\phi$. Let I be the ideal corresponding to the vanishing of s. Then the top dimensional part of I is arithmetically Gorenstein of codimension r. Denoting by J this top dimensional part, the minimal free resolution of R/J can be written in terms of F and G.*

This can be used to find an arithmetically Gorenstein scheme (of the same dimension) containing a given one by means of the following corollary.

Corollary 4.19. *If* $\operatorname{codim} V = r$*, a regular section of $H^0_*(\mathbb{P}^n, \tilde{B}_\phi \otimes \mathcal{I}_V)$ has top dimensional part which is an arithmetically Gorenstein scheme, X, containing V.*

Theorem 4.18 is just a small sample (the application relevant to liaison) of the possible results on sections of Buchsbaum-Rim sheaves, and we refer the interested reader to [78] for more general results.

Our final construction requires a little preparation.

Definition 4.20. A subscheme $S \subset \mathbb{P}^n$ satisfies condition G_r if every localization of R/I_S of dimension $\leq r$ is a Gorenstein ring. G_r is sometimes referred to as "Gorenstein in codimension $\leq r$, i.e. the "bad locus" has codimension $\geq r+1$.

Definition 4.21. Let $S \subset \mathbb{P}^n$ be an arithmetically Cohen-Macaulay subscheme and let F be a homogeneous polynomial of degree d not vanishing on any component of S (i.e. $I_S : F = I_S$). Then H_F is the divisor on S cut out by F. We call H_F the *hypersurface* *section of S cut out by F.* As a subscheme of $\mathbb{P}^n$, H_F is defined by the ideal $I_S + (F)$. Note that this ideal is saturated, since S is arithmetically Cohen-Macaulay. (The idea is the same as that in Lemma 3.13 and Remark 3.14.)

Hartshorne [50] has developed the theory of divisors, and in particular linear equivalence, on schemes having at least G_1. Using the notion of linear equivalence, the following theorem gives our construction. In subsequent sections we will give some important applications for liaison.

Theorem 4.22 ([61])**. (Twisted anticanonical divisors)** *Let $S \subset \mathbb{P}^n$ be an arithmetically Cohen-Macaulay subscheme satisfying G_1 and let K be a canonical divisor of S. Then every effective divisor in the linear system $|dH - K|$, viewed as a subscheme of $\mathbb{P}^n$, is arithmetically Gorenstein.*

Proof. (Sketch)
Let $X \in |dH - K|$ be an effective divisor. Choose a sufficiently large integer ℓ such that there is a regular section of $\omega_S(\ell)$ defining a twisted canonical divisor Y. Let $F \in I_Y$ be a homogeneous polynomial of degree $d + \ell$ such that F does not vanish on any component of S and let H_F be the corresponding hypersurface section.

Then X is linearly equivalent to the effective divisor $H_F - Y$ and we have isomorphisms

$$(\mathcal{I}_X/\mathcal{I}_S)(d) \cong \mathcal{I}_{X|S}(d) \cong \mathcal{O}_S((d+\ell)H - X) \cong \mathcal{O}_S(Y) \cong \omega_S(\ell).$$

Because S is arithmetically Cohen-Macaulay, this gives

$$0 \to I_S \to I_X \to H^0_*(\omega_S)(\ell - d) \to 0.$$

Then considering a minimal free resolution of I_S and the corresponding one for $K_S = H^0_*(\omega_S)$ (cf. Remark 3.21) we have a diagram (ignoring twists)

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & R & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & F_c & & & & F_1^* & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & F_1 & & & & F_c^* & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \to & I_S & \to & I_X & \to & K_S & \to & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

Then the Horseshoe Lemma ([109] 2.2.8, p. 37) shows that I_X has a free resolution in which the last free module has rank one. Since $\operatorname{codim} X = c$, this last free module cannot split off, so X is arithmetically Gorenstein as claimed. $\square$

Example 4.23. Let S be a twisted cubic curve in $\mathbb{P}^3$. Then a canonical divisor K has degree -2, so the linear system $|-K + dH|$ (for $d \geq 0$) consists of all effective divisors of degree $\equiv 2 \pmod 3$. Any such scheme is arithmetically Gorenstein.

5. First relations between linked schemes

In this section we begin to investigate the relations between linked ideals. In particular, we will compare the Hilbert functions of directly linked ideals and cover some of the results announced in Section 2.

All the ideals will be homogeneous ideals of the polynomial ring $R = K[x_0, \dots, x_n]$. Analogously to linked schemes we define linked ideals. This includes linkage of Artinian ideals corresponding to empty schemes.

Definition 5.1. (i) Two unmixed ideals $I, J \subset R$ of the same codimension are said to be *geometrically CI-linked* (resp. *geometrically G-linked*) by the ideal $\mathfrak{c}$ if I and J do not have associated prime ideals in common and $\mathfrak{c} = I \cap J$ is a complete intersection (resp. a Gorenstein ideal).

(ii) Two ideals $I, J \subset R$ are said to be *(directly) CI-linked* (resp. *(directly) G-linked*) by the ideal $\mathfrak{c}$ if $\mathfrak{c}$ is a complete intersection (resp. a Gorenstein ideal) and

$$\mathfrak{c} : I = J \quad \text{and} \quad \mathfrak{c} : J = I.$$

In this case we write $I \overset{\mathfrak{c}}{\sim} J$.

If a statement is true for CI-linked ideals and G-linked ideals we will just speak of linked ideals.

Remark 5.2. (i) If we want to stress the difference between (i) and (ii) we say in case (ii) that the ideals are algebraically linked.

(ii) If two ideals are geometrically linked then they are also algebraically linked.

(iii) Since Gorenstein ideals of codimension two are complete intersections CI-linkage is the same as G-linkage for ideals of codimension two.

If the subschemes V and W are geometrically linked by X then $\deg X = \deg V + \deg W$. We will see that this equality is also true if V and W are only algebraically linked. For this discussion we will use the following.

Notation 5.3. $I, \mathfrak{c} \subset R$ denote homogeneous ideals where $\mathfrak{c}$ is a Gorenstein ideal of codimension c. Excluding only trivial cases we assume $c \geq 2$.

Lemma 5.4. *If $I \nsubseteq \mathfrak{c}$ then $\mathfrak{c} : I$ is an unmixed ideal of codimension c.*

Proof. Let $\mathfrak{c} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s$ be a shortest primary decomposition of $\mathfrak{c}$. Then the claim follows because

$$\mathfrak{c} : I = \bigcap_{i=1}^{s} (\mathfrak{q}_i : I)$$

and

$$\mathfrak{q}_i : I = \begin{cases} R & \text{if } I \subset \mathfrak{q}_i \\ \operatorname{Rad}(\mathfrak{q}_i) - \text{primary} & \text{otherwise.} \end{cases}$$

□

The next observation deals with the difference between geometric and algebraic linkage.

Corollary 5.5. *Suppose the ideals I and J are directly linked by $\mathfrak{c}$. Then we have:*

(a) $\operatorname{Rad}(I \cap J) = \operatorname{Rad} \mathfrak{c}$.

(b) *I and J are unmixed of codimension c.*
(c) *If I and J do not have associated prime ideals in common then I and J are geometrically linked by $\mathfrak{c}$*

Proof. (a) By definition we have

$$I \cdot J \subset \mathfrak{c} \subset I \cap J.$$

Since $\operatorname{Rad}(I \cdot J) = \operatorname{Rad}(I \cap J)$ the claim follows.

(b) is an immediate consequence of the preceding lemma.

(c) We have to show that $I \cap J = \mathfrak{c}$.

The inclusion $\mathfrak{c} \subset I \cap J$ is clear. For showing the other inclusion assume on the contrary that there is a homogeneous polynomial $f \neq 0$ in $(I \cap J) \backslash \mathfrak{c}$. Let $\mathfrak{c} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s$ be a shortest primary decomposition of $\mathfrak{c}$. We may assume that $f \notin \mathfrak{q}_1$ and $\operatorname{Rad} \mathfrak{q}_1 \in \operatorname{Ass}_R(R/I)$. The assumption on the associated prime ideals of I and J guarantees that there is a homogeneous polynomial $g \in J$ such that $g \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R(R/I)$. Since $I = \mathfrak{c} : J$ we get $fg \in \mathfrak{c} \subset \mathfrak{q}_1$. Thus, $g \notin \operatorname{Rad} \mathfrak{q}_1$ implies $f \in \mathfrak{q}_1$, a contradiction. □

Claim (a) of the statement above says for schemes V, W linked by X that we have (as sets) $V_{red} \cup W_{red} = X_{red}$.

In order to identify certain degree shifts we need the following number. It is well-defined because the Hilbert function equals the Hilbert polynomial in all sufficiently large degrees.

Definition 5.6. The *regularity index* of a finitely generated graded R-module M is the number

$$r(M) := \min\{t \in \mathbb{Z} \mid h_M(j) = p_M(j) \text{ for all } j \geq t\}.$$

Example 5.7. (i) $r(K[x_0, \ldots, x_n]) = -n$.

(ii) If A is an Artinian graded K-algebra with $s = \max\{j \in \mathbb{Z} \mid [A]_j \neq 0\}$ then $r(A) = s + 1$.

(iii) Let $\mathfrak{c} \in R$ be a Gorenstein ideal with minimal free resolution

$$0 \to R(-t) \to F_{c-1} \to \ldots \to F_1 \to \mathfrak{c} \to 0.$$

Then it is not to difficult to see that $r(R/\mathfrak{c}) = t - n$ (cf. also Corollary 4.7).

The index of regularity should not be confused with the Castelnuovo-Mumford regularity. There is the following comparison result.

Lemma 5.8. *Let M be a graded R-module. Then we have*

$$\operatorname{reg} M - \dim M + 1 \leq r(M) \leq \operatorname{reg} M - \operatorname{depth} M + 1.$$

In particular, $r(M) = regM - \dim M + 1$ if M is Cohen-Macaulay.

This lemma generalizes Example 5.7(ii) and is a consequence of the following version of the Riemann-Roch theorem [102] which we will use again soon.

Lemma 5.9. *Let M be a graded R-module. Then we have for all $j \in \mathbb{Z}$*

$$h_M(j) - p_M(j) = \sum_{i \geq 0} (-1)^i \operatorname{rank}_K [H^i_{\mathfrak{m}}(M)]_j.$$

We are now ready for a crucial observation.

Lemma 5.10 (Standard exact sequences). *Suppose that $\mathfrak{c} \subset I$ and both ideals have the same codimension c. Put $J := \mathfrak{c} : I$. Then there are exact sequences (of graded R-modules)*

$$0 \to \mathfrak{c} \to J \to K_{R/I}(1 - r(R/\mathfrak{c})) \to 0$$

and

$$0 \to K_{R/I}(1 - r(R/\mathfrak{c})) \to R/\mathfrak{c} \to R/J \to 0.$$

Proof. We have to show that $J/\mathfrak{c} \cong K_{R/I}(1 - r(R/\mathfrak{c}))$.

There are the following isomorphisms

$$J/\mathfrak{c} \cong (\mathfrak{c} : I)/\mathfrak{c} \cong \operatorname{Hom}_R(R/I, R/\mathfrak{c}) \cong \operatorname{Hom}_{R/\mathfrak{c}}(R/I, R/\mathfrak{c})$$

and

$$K_{R/I}(1 - r(R/\mathfrak{c})) \cong \operatorname{Ext}^c_R(R/I, R)(-r(R/\mathfrak{c}) - n).$$

Thus, our claim follows from the isomorphism

$$\operatorname{Hom}_R(R/I, R/\mathfrak{c}) \cong \operatorname{Ext}^c_R(R/I, R)(-r(R/\mathfrak{c}) - n),$$

which is easy to see if $\mathfrak{c}$ is a complete intersection. In the general case it follows from a more abstract characterization of the canonical module. □

Before drawing first consequences we recall that the Hilbert polynomial of the graded module M can be written in the form

$$p_M(j) = h_0(M)\binom{j}{d-1} + h_1(M)\binom{j}{d-2} + \ldots + h_{d-1}(M)$$

where $d = \dim M$ and $h_0(M), \ldots, h_{d-1}(M)$ are integers. Moreover, if $d > 0$ then $\deg M = h_0(M)$ is positive and called the degree of M. However, if $M = R/I$ for an ideal I then by abuse of notation we define $\deg I := \deg R/I$. For a subscheme $X \subset \mathbb{P}^n$ we have then $\deg X = \deg I_X$.

Corollary 5.11. *Let I be an ideal of codimension c which contains the Gorenstein ideal $\mathfrak{c}$. Put $J := \mathfrak{c} : I$ and $s := r(R/\mathfrak{c}) - 1$. Then we have*

(a)

$$\deg J = \deg \mathfrak{c} - \deg I,$$

and if $c < n$ and I is unmixed then

$$h_1(R/J) = \frac{1}{2}(s - n + c + 1)[\deg I - \deg J] + h_1(R/I).$$

(b) *If I is unmixed and R/I is locally Cohen-Macaulay then also R/J is locally Cohen-Macaulay and*

$$H^i_\mathfrak{m}(R/J) \cong H^{n+1-c-i}_\mathfrak{m}(R/I)^\vee(-s) \quad (i = 1, \ldots, n-c)$$

and

$$p_{R/J}(j) = p_{R/\mathfrak{c}}(j) + (-1)^{n-c} p_{R/I}(s-j).$$

(c) *If R/I is Cohen-Macaulay then also R/J has this property and*

$$h_{R/J}(j) = h_{R/\mathfrak{c}}(j) + (-1)^{n-c} h_{R/I}(s-j).$$

Proof. Consider the version of Riemann-Roch (Lemma 5.9)

$$h_{R/I}(j) - p_{R/I}(j) = \sum_{i \geq 0} (-1)^i \operatorname{rank}_K [H^i_{\mathfrak{m}}(R/I)]_j.$$

Since the degree of the Hilbert polynomial of $H^i_{\mathfrak{m}}(R/I)$ is at most $i-1$, we obtain for all $j \ll 0$

$$\begin{aligned} -p_{R/I}(j) &= (-1)^{n+1-c} \operatorname{rank}_K [H^{n+1-c}_{\mathfrak{m}}(R/I)]_j + O(j^{n-1-c}) \\ &= (-1)^{n+1-c} \operatorname{rank}_K [K_{R/I}]_{-j} + O(j^{n-1-c}). \end{aligned}$$

Combined with the standard exact sequence this provides

$$p_{R/J}(j) = p_{R/\mathfrak{c}}(j) + (-1)^{n-c} p_{R/I}(s-j) + O(j^{n-1-c}).$$

Comparing coefficients we get if $c \leq n$

$$\deg J = \deg \mathfrak{c} - \deg I.$$

Assume now that the ideal I is unmixed. Lemma 3.11 implies that then the degree of the Hilbert polynomial of $H^i_{\mathfrak{m}}(R/I)$ is at most $i-2$. Thus, we obtain as above

$$p_{R/J}S(j) = p_{R/\mathfrak{c}}(j) + (-1)^{n-c} p_{R/I}(s-j) + O(j^{n-2-c}).$$

Hence, we get if $c < n$

$$h_1(R/J) = (s-n+1+c) \deg I + h_1(R/I) + h_1(R/\mathfrak{c}).$$

But by duality we have

$$h_1(R/\mathfrak{c}) = (s-n+1+c) \deg \mathfrak{c}.$$

Combining the last two equalities proves the second statement in (a).

The isomorphisms

$$H^i_{\mathfrak{m}}(R/J) \cong H^{n+1-i}_{\mathfrak{m}}(R/I)^{\vee}(-s) \quad (i = 1. \dots, n-c)$$

follow essentially from the long exact cohomology sequence induced by the standard sequence taking into account Proposition 3.22.

The remaining claims in (a) - (c) are proved similarly as above. For details we refer to [85]. □

Remark 5.12. If I is unmixed but not locally Cohen-Macaulay then the formula in claim (b) relating the local cohomologies of I and J is not true in general. For example, it is never true if I defines a non-locally Cohen-Macaulay surface in $\mathbb{P}^4$.

The last statement applies in particular to directly linked ideals. Thus, we obtain.

Corollary 5.13. *Let V and W be directly linked. Then we have:*

(a) *V is arithmetically Cohen-Macaulay if and only if W is arithmetically Cohen-Macaulay.*
(b) *V is locally Cohen-Macaulay if and only if W has this property.*

Before turning to examples we want to rewrite Claim (a) in Corollary 5.11 for curves in a more familiar form.

Remark 5.14. Let $C_1, C_2 \subset \mathbb{P}^n$ be curves linked by an arithmetically Gorenstein subscheme X. Let g_1 and g_2 denote the arithmetic genus of C_1 and C_2, respectively. Since by definition $g_i = 1 - h_1(R/I_{X_i})$, Corollary 5.11 provides the formula

$$g_1 - g_2 = \frac{1}{2}(r(R/I_X) - 1) \cdot [\deg C_1 - \deg C_2].$$

In particular, if X is a complete intersection cut out by hypersurfaces of degree $d_1, \dots, d_{n-1}$ we obtain

$$g_1 - g_2 = \frac{1}{2}(d_1 + \ldots + d_{n-1} - n - 1) \cdot [\deg C_1 - \deg C_2]$$

because the index of regularity of X is

$$r(R/I_X) = d_1 + \ldots + d_{n-1} - n.$$

Example 5.15. (i) Let $C \subset \mathbb{P}^3$ be the twisted cubic parameterized by (s^3, s^2t, st^2, t^3). It is easy to see that C is contained in the complete intersection X defined by

$$\mathfrak{c} := (x_0x_3 - x_1x_2, x_0x_2 - x_1^2) \subset I_C.$$

Then Corollary 5.11 shows that $\mathfrak{c} : I_C$ has degree 1. In fact, we easily get $\mathfrak{c} : I_C = (x_0, x_1)$ defining the line L. Thus, C and L are geometrically linked by X and C is arithmetically Cohen-Macaulay.

(ii) Let $C \subset \mathbb{P}^3$ be the rational quartic parameterized by (s^4, s^3t, st^3, t^4). C is contained in the complete intersection X defined by

$$\mathfrak{c} := (x_0x_3 - x_1x_2, x_0x_2^2 - x_1^2x_3) \subset I_C.$$

Hence $\mathfrak{c} : I_C$ has degree 2. Indeed, it is easy to see that

$$\mathfrak{c} : I_C = (x_0, x_1) \cap (x_2, x_3).$$

This implies that C is geometrically linked to a pair of skew lines. Therefore C is not arithmetically Cohen-Macaulay (thanks to Example 3.28).

(iii) We want to illustrate Corollary 5.11 (c). Let $I := (x^2, xy, y^4) \subset R := K[x, y]$ and let $\mathfrak{c} := (x^3, y^4) \subset I$. We want to compute the Hilbert function of R/J. Consider the following table

j	0	1	2	3	4	5	6
$h_{R/I}(j)$	1	2	1	1	0	0	0
$h_{R/\mathfrak{c}}(j)$	1	2	3	3	2	1	0
$h_{R/J}(5-j)$	0	0	2	2	2	1	0

The second row shows that $r(R/\mathfrak{c}) = 6$. Thus by Corollary 5.11 (c), the last row is the second row minus the first row. We get that R/J has the Hilbert function $1, 2, 2, 2, 0, \ldots$.

We justify now Remark 2.9.

Corollary 5.16. *Suppose I is an unmixed ideal of codimension c. If the Gorenstein ideal $\mathfrak{c}$ is properly contained in I then the ideals $\mathfrak{c} : I$ and I are directly linked by $\mathfrak{c}$.*

Proof. We have to show the equality

$$\mathfrak{c} : (\mathfrak{c} : I) = I.$$

It is clear that $I \subset \mathfrak{c} : (\mathfrak{c} : I)$. Corollary 5.11 (a) provides

$$\deg[\mathfrak{c} : (\mathfrak{c} : I)] = \deg \mathfrak{c} - \deg(\mathfrak{c} : I) = \deg I.$$

Since $\mathfrak{c} : (\mathfrak{c} : I)$ and I are unmixed ideals of the same codimension they must be equal. □

Finally, we want to show how CI-linkage can be used to produce Gorenstein ideals. To this end we introduce.

Definition 5.17. An ideal $I \subset R$ is called an *almost complete intersection* if R/I is Cohen-Macaulay and I can be generated by $\operatorname{codim} I + 1$ elements.

Example 5.18. The ideal $(x_0, x_1)^2$ is an almost complete intersection.
The twisted cubic $C \subset \mathbb{P}^3$ is also an almost complete intersection.

Corollary 5.19. *Let $I \subset R$ be an almost complete intersection and let $\mathfrak{c} \subsetneq I$ be a complete intersection such that* $\operatorname{codim} I = \operatorname{codim} \mathfrak{c}$ *and $I = \mathfrak{c} + fR$ for some $f \in R$. Then $J := \mathfrak{c} : I$ is a Gorenstein ideal.*

Proof. Consider the standard exact sequence

$$0 \to \mathfrak{c} \to \mathfrak{c} + fR \to K_{R/J}(1 - r(R/\mathfrak{c})) \to 0.$$

It shows that $K_{R/J}$ has just one minimal generator (as R-module).
Let

$$0 \to F_c \xrightarrow{\varphi_c} \ldots \to F_1 \to J \to 0$$

be a minimal free resolution. Then the beginning of the minimal free resolution of $K_{R/J}$ has the form

$$\ldots \to F_{c-1}^* \xrightarrow{\varphi_c^*} F_c^* \to K_{R/J}(n+1) \to 0.$$

It follows that F_c must have rank 1, i.e. J is a Gorenstein ideal. □

Remark 5.20. Every Gorenstein ideal arises as in the previous corollary.
We only sketch the argument. Given a Gorenstein J of codimension c we choose a complete intersection $\mathfrak{c}$ of codimension c which is properly contained in J. Then $I := \mathfrak{c} : J$ is an almost complete intersection and $J = \mathfrak{c} : I$.

6. Some Basic Results and Constructions

We begin this section by proving one of the results mentioned in Remark 2.14.

Theorem 6.1. *Let $S \subset \mathbb{P}^n$ be arithmetically Cohen-Macaulay satisfying property G_1 (so that linear equivalence is well-defined; see Definition 4.20 and the discussion preceding Theorem 4.22). Let $C_1, C_2 \subset S$ be divisors such that $C_2 \in |C_1 + tH|$, where H is the class of a hyperplane section and $t \in \mathbb{Z}$. Then C_1 and C_2 are G-linked in two steps.*

Proof. Let Y be an effective twisted canonical divisor. Choose an integer $a \in \mathbb{Z}$ such that $[I_Y]_a$ contains a form A not vanishing on any component of S. Hence $H_A - Y$ is effective on S.

Now, recall that

$$\begin{array}{rcl} C_2 \in |C_1| & \Leftrightarrow & C_2 - C_1 \text{ is the divisor of a rational function on } S \\ & \Leftrightarrow & \text{there exist } F, G \text{ of the same degree such that } \left(\frac{F}{G}\right) = C_2 - C_1 \\ & & \text{(where } \left(\frac{F}{G}\right) \text{ is the divisor of the rational function } \frac{F}{G}\text{)} \\ & \Leftrightarrow & \text{there exists a divisor } D \text{ such that} \\ & & \qquad H_F = C_2 + D \\ & & \qquad H_G = C_1 + D \\ & & \text{(in particular, } F \in I_{C_2} \text{ and } G \in I_{C_1}\text{).} \end{array}$$

Similarly,

$$\begin{array}{rcl} C_2 \in |C_1 + tH| & \Leftrightarrow & \text{there exist } F, G \text{ with } \deg F = \deg G + t \text{ and a divisor } D \text{ such that} \\ & & \qquad H_F = C_2 + D \\ & & \qquad H_G = C_1 + D \\ & & \text{(in particular, } F \in I_{C_2} \text{ and } G \in I_{C_1}\text{).} \end{array}$$

Note that the effective divisor $H_{AF} - Y$ is arithmetically Gorenstein, by Theorem 4.22. Then one checks if S is smooth that

$$(H_{AF} - Y) - C_2 = (H_A - Y) + (H_F - C_2) = (H_A - Y) + D$$

and

$$(H_{AG} - Y) - C_1 = (H_A - Y) + (H_G - C_1) = (H_A - Y) + D$$

Therefore C_2 is directly linked to $(H_A - Y) + D$ by the Gorenstein ideal $H_{AF} - Y$ and C_1 is directly linked to the same $(H_A - Y) + D$ by the Gorenstein ideal $H_{AG} - Y$. This concludes the proof in the special case. For the general case we refer to [61], Proposition 5.12. ☐

Theorem 6.1 was the first result that really showed that Gorenstein liaison is a theory about divisors on arithmetically Cohen-Macaulay schemes, just as Hartshorne [50] had shown that CI-liaison is a theory about divisors on complete intersections. It is fair to say that most of the results about Gorenstein liaison discovered in the last few years use this result either directly or at least indirectly.

Remark 6.2. As pointed out to us by R. Hartshorne, there is an interesting point lurking in the background here. Following [68] and [50], we say that a subscheme $V_2 \subset \mathbb{P}^n$ is obtained from a subscheme $V_1 \subset \mathbb{P}^n$ by an *elementary CI-biliaison* if there is a complete intersection S in $\mathbb{P}^n$ such that $V_2 \sim V_1 + hH$ on S for some integer $h \geq 0$, where $\sim$ denotes linear equivalence. It is not hard to show, and has long been known, that V_1 and V_2 are CI-linked in two steps. It is a theorem ([50]) that the equivalence relation generated by elementary CI-biliaisons is the same as the equivalence relation of even CI-liaison (see Definition 7.1).

Now, Theorem 6.1 naturally suggests the idea of saying that V_2 is obtained from V_1 by an *elementary G-biliaison* if there is an arithmetically Cohen-Macaulay scheme S with property G_1 such that $V_2 \sim V_1 + hH$ on S for some integer $h \geq 0$. It is an open problem to determine if the equivalence relation generated by elementary G-biliaisons is the same as the equivalence relation of even G-liaison. It is conceivable that schemes V_1 and V_2 could be evenly G-linked, but no sequence of elementary G-biliaisons beginning with V_1 can arrive at V_2.

As mentioned above, many of the results about Gorenstein liaison in fact use elementary G-biliaisons, so the results are actually slightly stronger in this sense.

Theorem 6.1 clearly needs the G_1 assumption since linear equivalence was used. In general, without G_1, it is not always possible to talk about divisors of the form $H_A - Y$. However, we will see now that there is a notion of "adding" a hypersurface section even if the G_1 assumption is relaxed. This construction was given in Lemma 3.25 and was called "Basic Double Link" there. Now we will see why this name was chosen. In Lemma 3.25 almost no assumption was made on the ideal J. Here we present it in more geometric language, and we have to assume at least G_0 in order to get a liaison result.

Proposition 6.3 (Basic Double G-Linkage). *Let $S \subset \mathbb{P}^n$ be an arithmetically Cohen-Macaulay subscheme satisfying G_0. Let $C \subset S$ be an equidimensional subscheme of codimension 1 and let $A \in R$ be homogeneous with $I_S : A = I_S$ (i.e. A does not vanish on any component of S). Then I_C and $I_S + A \cdot I_C$ are G-linked in two steps.*

Remark 6.4. As we will see, the ideal $I_S + A \cdot I_C$ represents the divisor $C + H_A$ on S. If $\deg A = d$ and S satisfies G_1 then the scheme Y defined by the ideal $I_Y + I_S + A \cdot I_C$ is in the linear system $|C + dH|$. But in our level of generality, linear systems may not make sense. See [73] for a more detailed discussion of these divisors.

Proof of Proposition 6.3 (sketch)

The unmixedness statement in Lemma 3.25 shows that in particular, $I_Y = I_S + A \cdot I_C$ is saturated. Furthermore, from the exact sequence (3.1) we can make a Hilbert function calculation:

$$h_{R/I_Y}(t) = h_{R/I_S}(t) - h_{R/I_S}(t-d) + h_{R/I_C}(t-d).$$

It follows that

$$\begin{aligned} \deg Y &= \deg C + d \cdot \deg S \\ &= \deg C + \deg H_A. \end{aligned}$$

The idea of the proof is to mimic the proof of Theorem 6.1 in an algebraic way. We proceed in four steps:

Step I: Let $c = \operatorname{codim} S$. The G_0 hypothesis is enough to guarantee that there exists a *Gorenstein* ideal $J \subset R$ with $I_S \subset J$, $\operatorname{codim} J = c + 1$ and J/I_S is Cohen-Macaulay of Cohen-Macaulay type 1 (cf. [61]). Since $\operatorname{codim} C > \operatorname{codim} S$, there exists $B \in I_C$ of some degree, e, such that $I_S : B = I_S$ (i.e. B does not vanish on any component of S).

Step II: One checks that $I_S + B \cdot J$ is Gorenstein and $I_S + B \cdot J \subset I_C$. Hence $I_S + B \cdot J$ links I_C to some ideal $\mathfrak{a}$ which is unmixed.

Step III: $I_S + AB \cdot J$ is Gorenstein and is contained in $I_Y = I_S + A \cdot I_C$. Hence $I_S + AB \cdot J$ links I_Y to some ideal $\mathfrak{b}$ which is unmixed.

<u>*Step IV*</u>: One can check that $\mathfrak{a} \subset \mathfrak{b}$ and compute that $\deg \mathfrak{a} = \deg \mathfrak{b}$. Since both are unmixed of the same dimension, it follows that $\mathfrak{a} = \mathfrak{b}$. Hence I_C is G-linked to I_Y in two steps. □

Remark 6.5. A special case of Proposition 6.3 is worth mentioning. Suppose that S is a complete intersection, $I_S = (F_1, \dots, F_c)$, and $I_Y = A \cdot I_C + (F_1, \dots, F_c)$. Then all the links in Proposition 6.3 are complete intersections. This construction is called *Basic Double CI-Linkage*; cf. [63], [18], [43]. As an even more special case, suppose that $c = 1$ and hence $\operatorname{codim} C = 2$. Let $F \in I_C$ and assume that A, F have no common factor. Then $I_Y = A \cdot I_C + (F)$. This construction is central to the Lazarsfeld-Rao property, which we will discuss below. This property is only known in codimension two.

A different way of viewing Basic Double Linkage, as a special case of Liaison Addition, will be discussed next.

Liaison Addition was part of the Ph.D. thesis of Phil Schwartau [101]. The problem which he originally considered was the following. Consider curves $C_1, C_2 \subset \mathbb{P}^3$. Suppose that $H^1_*(\mathcal{I}_{C_1}) = M_1$ and $H^1_*(\mathcal{I}_{C_2}) = M_2$. Find an explicit construction of a curve C for which $H^1_*(\mathcal{I}_C) = M_1 \oplus M_2$.

The first observation to make is that this is impossible in general!! We give a simple example.

Example 6.6. Let C_1 and C_2 be two disjoint sets of two skew lines. We have noticed (Example 3.28) that $H^1_*(\mathcal{I}_{C_1}) = H^1_*(\mathcal{I}_{C_2}) = K$, a graded module of dimension 1 occurring in degree 0. So the question is whether there exists a curve C with $H^1(\mathcal{I}_C) = K^2$, a 2-dimensional module occurring in degree 0. Suppose that such a curve exists. Let L be a general linear form defining a hyperplane H. We have the long exact sequence

$$\begin{array}{ccccccccccc} 0 & \to & H^0(\mathcal{I}_C) & \to & H^0(\mathcal{I}_C(1)) & \to & H^0(\mathcal{I}_{C\cap H|H}(1)) & \to & H^1(\mathcal{I}_C) & \to & H^1(\mathcal{I}_C(1)) & \to \dots \\ & & || & & || & & & & || & & || & \\ & & 0 & & 0 & & & & 2 & & 0 & \end{array}$$

Note that $h^0(\mathcal{I}_C(1)) = 0$ since otherwise C would be a plane curve, hence a hypersurface and hence arithmetically Cohen-Macaulay. But this means that C is a curve whose general hyperplane section lies on a pencil of lines in $\mathbb{P}^2$. This forces $\deg C = 1$, hence C is a line and is thus arithmetically Cohen-Macaulay. Contradiction.

However, an important idea that we have seen in Section 3 is that the *shift* of the modules is of central importance. Hence the refined problem that Schwartau considered is whether there is a construction of a curve C for which $H^1_*(\mathcal{I}_C) = M_1 \oplus M_2$ *up to shift.* As we will see, he was able to answer this question and even a stronger one (allowing the modules to individually have different shifts), and his work was for codimension two in general. The version that we will give is a more general one, however, from [43]. The statement, but not the proof, were inspired by [101], which proved the case $r = 2$.

Theorem 6.7. *Let $V_1, \dots, V_r$ be closed subschemes of $\mathbb{P}^n$, with $2 \le r \le n$. Assume that* $\operatorname{codim} V_i \ge r$ *for all i. Choose homogeneous polynomials*

$$F_i \in \bigcap_{\substack{1 \le j \le r \\ j \ne i}} I_{V_j} \qquad \text{for } 1 \le i \le r$$

such that $(F_1, \dots, F_r)$ form a regular sequence, hence defining a complete intersection, $V \subset \mathbb{P}^n$. Let $d_i = \deg F_i$. Define the ideal $I = F_1 \cdot I_{V_1} + \cdots + F_r \cdot I_{V_r}$. Let Z be the closed subscheme of $\mathbb{P}^n$ defined by I (which a priori is not saturated). Then

(a) *As sets, $Z = V_1 \cup \cdots \cup V_r \cup V$;*
(b) *For all $1 \le j \le n - r = \dim V$ we have*

$$H^j_*(\mathcal{I}_Z) \cong H^j_*(\mathcal{I}_{V_1})(-d_1) \oplus \cdots \oplus H^j_*(\mathcal{I}_{V_r})(-d_r);$$

(c) *I is saturated.*

We will not give the proof of this theorem, but refer the reader to [43] or to [72].

Remark 6.8. Note that we allow $V_1, \dots, V_r$ to be of different codimensions, we allow them to fail to be locally Cohen-Macaulay or equidimensional, and we even allow them to be empty. In fact, this latter possibility gives another approach to Basic Double CI-Linkage (cf. Remark 6.5). Indeed, if we let $V_2 = \cdots = V_r = \emptyset$, with $I_{V_2} = \cdots = I_{V_r} = R$, and set $V_1 = C$ and $V = S$ as in Remark 6.5, then the ideal

$$\begin{aligned} I &= F_1 \cdot I_{V_1} + \cdots + F_r \cdot I_{V_r} \\ &= F_1 \cdot I_{V_1} + (F_2, \dots, F_r) \end{aligned}$$

is precisely the ideal of the Basic Double CI-Linkage.

An application of Liaison Addition is to the construction of arithmetically Buchsbaum curves, or more generally arithmetically Buchsbaum subschemes of projective space. We will give the basic idea here and come back to it in Section 11.

Definition 6.9. A curve $C \subset \mathbb{P}^n$ is *arithmetically Buchsbaum* if $H^1_*(\mathcal{I}_C)$ is annihilated by the "maximal ideal" $\mathfrak{m} = (x_0, \dots, x_n)$. A subscheme $V \subset \mathbb{P}^n$ of dimension $d \ge 2$ is arithmetically Buchsbaum if $H^i_*(\mathcal{I}_V)$ is annihilated by $\mathfrak{m}$ for $1 \le i \le d$ and furthermore the general hyperplane section $V \cap H$ is an arithmetically Buchsbaum scheme in $H = \mathbb{P}^{n-1}$.

Buchsbaum curves in $\mathbb{P}^3$, especially, are fascinating objects which have been studied extensively. A rather large list of references can be found in [72]. Liaison Addition can be used to construct examples of arithmetically Buchsbaum curves with modules whose components have any prescribed dimensions, up to shift. Indeed, Schwartau's original work [101] already produced examples of modules of any dimension, concentrated in one degree. The more general result was obtained in [17]. The idea is to use sets of two skew lines as a "building block" to build bigger curves. We will give the basic idea with an example, omitting the proof of the general result.

Example 6.10. Recall (Example 3.28) that the deficiency module of a set of two skew lines is one-dimensional as a K-vector space, and the non-zero component occurs in degree 0. Furthermore, this is the minimal shift for that module. Let C_1 and C_2 be two such curves. How C_1 and C_2 meet (i.e. whether they are disjoint from each other, meet in finitely many points or contain common components) is not important. Let $F_1 \in I_{C_2}$ and $F_2 \in I_{C_1}$ such that F_1 and F_2 have no common factor. Note that $\deg F_i \geq 2$. Then the curve C obtained by $I_C = F_1 \cdot I_{C_1} + F_2 \cdot I_{C_2}$ is arithmetically Buchsbaum and its deficiency module is the direct sum of twists of the deficiency modules of C_1 and C_2. In particular, this module is 2-dimensional as a K-vector space. The components occur in degrees $\deg F_1$ and $\deg F_2$. In particular, the components can be arbitrarily far apart, and regardless of how far apart they are, the leftmost component occurs in degree ≥ 2. Furthermore, an example can be obtained for which this component is in degree exactly 2 (for instance by choosing F_1 with $\deg F_1 = 2$ and then F_2 of appropriate degree). Note that $2 = 2 \cdot 2 - 2$ (see Proposition 6.11).

An arithmetically Buchsbaum curve whose module is 3-dimensional as a K-vector space can then be constructed by taking the Liaison Addition of C with another set of two skew lines, and it is clear that this process can be extended to produce any module (up to shift) which is annihilated by $\mathfrak{m}$. Furthermore, if we produce in this way a curve whose deficiency module has dimension N as a K-vector space, a little thought shows that this can be done in such a way that the leftmost non-zero component occurs in degree $\geq 2N-2$, and that a sharp example can be constructed. We refer to [17] for details.

This approach was also used in [43] to construct arithmetically Buchsbaum curves in $\mathbb{P}^4$, and in [19] to construct certain arithmetically Buchsbaum surfaces in $\mathbb{P}^4$ with nice properties.

One would like to have an idea of "how many" of the Buchsbaum curves can be constructed using this approach together with Basic Double Linkage (which preserves the module but shifts it to the right, adding a complete intersection to the curve). The first step was obtained in [41]:

Proposition 6.11. *Let $C \subset \mathbb{P}^3$ be an arithmetically Buchsbaum curve. Let*

$$N = \dim H^1_*(\mathcal{I}_C) = \sum_{t \in \mathbb{Z}} h^i(\mathcal{I}_C(t)).$$

Then the first non-zero component of $H^1_*(\mathcal{I}_C)$ occurs in degree $\geq 2N-2$.

Proof. The proof is an easy application of a result of Amasaki ([1], [42]) which says that C lies on no surface of degree $< 2N$. We refer to [41] for the details. ☐

It follows that the construction of Example 6.10 provides curves which are in the minimal shift for their module. We will see below that the Lazarsfeld-Rao property then gives an incredible amount of information about all arithmetically Buchsbaum curves, once we know even one curve in the minimal shift. It will turn out that this construction together with Basic Double Linkage, gives *all* arithmetically Buchsbaum curves in $\mathbb{P}^3$ up to deformation. Furthermore, this construction will even give us information about arithmetically Buchsbaum stick figures.

7. Necessary conditions for being linked

Since (direct) linkage is symmetric, the transitive closure of this relation generates an equivalence relation, called liaison. However, it will be useful to study slightly different equivalence classes.

As in the previous sections we will restrict ourselves to subschemes of $\mathbb{P}^n$ and ideals of $R = K[x_0, \dots, x_n]$, although the results are more generally true for subschemes of an arithmetically Gorenstein scheme.

Definition 7.1. Let $I \subset R$ denote an unmixed ideal. Then the *even G-liaison class* of I is the set

$$\mathcal{L}_I = \{J \subset R \mid I \stackrel{\mathfrak{c}_1}{\sim} I_1 \stackrel{\mathfrak{c}_2}{\sim} \dots \stackrel{\mathfrak{c}_{2k}}{\sim} J\}$$

where $\mathfrak{c}_1, \dots, \mathfrak{c}_{2k}$ are Gorenstein ideals. If we require that all ideals $\mathfrak{c}_1, \dots, \mathfrak{c}_{2k}$ are complete intersections that we get the *even CI-liaison class* of I.

The even G-liaison class and the even CI-liaison class of an equidimensional subscheme $V \subset \mathbb{P}^n$ are defined analogously.

Remark 7.2. It is clear from the definition that every G-liaison class consists of at most two even G-liaison classes and that every CI-liaison class consists of at most two even CI-liaison classes.

A liaison class can agree with an even liaison class, for example, if it contains a self-linked element. On the other hand, it was shown by Rao [96] that there are liaison classes that coincide with even liaison classes but contain no self-linked elements, the simplest being the liaison class of two skew lines in $\mathbb{P}^3$.

The next result has been shown in various levels of generality by Chiarli, Schenzel, Rao, Migliore.

Lemma 7.3. *If $V \subset \mathbb{P}^n$ is an equidimensional locally Cohen-Macaulay subscheme and $W \in \mathcal{L}_V$ then there is an integer t such that*

$$H^i_*(\mathcal{I}_W) \cong H^i_*(\mathcal{I}_V)(t) \quad \textit{for all } \ i = 1, \dots, \dim V.$$

Proof. This follows immediately from the comparison of the cohomology of directly linked locally Cohen-Macaulay subschemes (Corollary 5.11) because we have for every finitely generated graded R-module M that $(M^\vee)^\vee \cong M$. □

Our next goal is to show that there is a stronger result which is true even if V is not locally Cohen-Macaulay. For this we have to consider certain types of exact sequences. The names have been coined by Martin-Deschamps and Perrin [68].

Definition 7.4. Let $I \subset R$ be a homogeneous ideal of codimension $c \geq 2$.

(i) An *E-type resolution* of I is an exact sequence of finitely generated graded R-modules

$$0 \to E \to F_{c-1} \to \dots \to F_1 \to I \to 0$$

where the modules $F_1, \dots, F_{c-1}$ are free.

(ii) An *N-type resolution* of I is an exact sequence of finitely generated graded R-modules

$$0 \to G_c \to \dots \to G_2 \to N \to I \to 0$$

where $G_2, \dots, G_c$ are free R-modules and $H^i_{\mathfrak{m}}(N) = 0$ for all i with $n + 2 - c \leq i \leq n$.

Remark 7.5. (i) The existence of an E-type resolution is clear because it is just the beginning of a free resolution of I. For the existence of an N-type resolution we refer to [85]. However, we will see that for an unmixed ideal the existence follows by liaison.

(ii) It is easy to see that

$$H^i_{\mathfrak{m}}(N) \cong H^{i-1}_{\mathfrak{m}}(R/I) \quad \text{for all } i \leq n+1-c$$

and

$$H^i_{\mathfrak{m}}(E) \cong H^{i-c}_{\mathfrak{m}}(R/I) \quad \text{for all } i \leq n.$$

This shows that the modules E and N "store" the deficiency modules of R/I.

(iii) The sheafifications $\tilde{E}$ and $\tilde{N}$ are vector bundles if and only if I defines an equidimensional locally Cohen-Macaulay subscheme.

By definition, the module E is a c-syzygy. It is not to difficult to check that N must be a torsion-free module. However, if the ideal I is unmixed then these modules have better properties.

Lemma 7.6. *Using the notation of Definition 7.1 the following conditions are equivalent:*

(a) *The ideal I is unmixed.*
(b) *The module N is reflexive, i.e. the bilinear map $N \times N^* \to R, (m, \varphi) \mapsto \varphi(m)$, induces an isomorphism $N \to N^{**}$.*
(c) *The module E is a $(c+1)$-syzygy.*

Proof. This follows from the cohomological characterization of these concepts. ☐

The next result establishes the crucial fact that E-type and N-type resolutions are interchanged under direct linkage.

Proposition 7.7. *Let $I, J \subset R$ be homogeneous ideals of codimension c which are directly linked by $\mathfrak{c}$. Suppose I has resolutions of E- and N-type as in Definition 7.1. Let*

$$0 \to D_c \to \ldots \to D_1 \to \mathfrak{c} \to 0$$

be a minimal free resolution of $\mathfrak{c}$. Put $s = r(R/\mathfrak{c}) + n$. Then J has an N-type resolution

$$0 \to F_1^*(-s) \to D_{c-1} \oplus F_2^*(-s) \to \ldots \to D_2 \oplus F_{c-1}^*(-s) \to D_1 \oplus E^*(-s) \to J \to 0$$

and an E-type resolution

$$0 \to N^*(-s) \to D_{c-1} \oplus G_2^*(-s) \to \ldots \to D_1 \oplus G_c^*(-s) \to J \to 0.$$

Proof. We want to produce an N-type resolution of J. We proceed in several steps.

(I) Resolving E we get an exact sequence

$$\begin{tikzcd}
\ldots \arrow[r] & F_{c+1} \arrow[r, "\varphi_{c+1}"] & F_c \arrow[rr, "\varphi_c"] \arrow[dr] & & F_{c-1} \arrow[r] & \ldots \arrow[r] & F_1 \arrow[r, "\varphi_1"] & I \arrow[r] & 0 \\
& & & E \arrow[ur] \arrow[dr] & & & & & \\
& & 0 \arrow[ur] & & 0 & & & &
\end{tikzcd}$$

Dualizing with respect to R gives a complex

$$0 \to R \to F_1^* \to \ldots \to F_{c-1}^* \xrightarrow{\varphi_c^*} F_c^* \xrightarrow{\varphi_{c+1}^*} F_{c+1}^*$$

and an exact sequence

$$0 \to E^* \to F_c^* \xrightarrow{\varphi_{c+1}^*} F_{c+1}^*.$$

If follows $\ker \varphi_{c+1}^* \cong E^*$, thus $\operatorname{Ext}_R^c(R/I, R) \cong E^*/\operatorname{im} \varphi_c^*$. Moreover, we get by duality that

$$\ker \varphi_{i+1}^*/\operatorname{im} \varphi_i^* \cong \operatorname{Ext}_R^i(R/I, R) \cong H_{\mathfrak{m}}^{n+1-i}(R/I)^\vee(1 - r(R)) = 0 \quad \text{if } i < c$$

because $\dim R/I = n + 1 - c$. Therefore we can splice the two complexes above together and the resulting diagram

$$\begin{array}{ccccc}
0 \to R \to F_1^* \to \ldots \to F_{c-1}^* & & \longrightarrow & & E^* \to \operatorname{Ext}_R^c(R/I, R) \to 0 \\
& \searrow & & \nearrow & \\
& & \operatorname{im} \varphi_c^* & & \\
& \nearrow & & \searrow & \\
0 & & & & 0
\end{array}$$

is exact.

(II) The self-duality of the minimal free resolution of $R/\mathfrak{c}$ and Corollary 4.7 provide $D_c = R(-s)$ and $D_{c-i}^* \cong D_i(s)$ for all $i = 1, \ldots, c-1$.

(III) Put $r := r(R/\mathfrak{c}) - 1$. The standard exact sequence provides the following diagram with exact rows and column:

$$\begin{array}{rcl}
 & 0 & \\
 & \downarrow & \\
 & K_{R/J}(-r) & \\
 & \downarrow & \\
0 \to R(-s) \to D_{c-1} \to \ldots \to D_1 \to R \to & R/\mathfrak{c} & \to 0 \\
 & \downarrow & \\
0 \to E \to F_{c-1} \to \ldots \to F_1 \to R \to & R/I & \to 0 \\
 & \downarrow & \\
 & 0. &
\end{array}$$

Since the modules $D_1, \ldots, D_{c-1}$ are free the epimorphism $R/\mathfrak{c} \to R/I$ lifts to a morphism of complexes. Thus, using steps (I) and (II) we get by dualizing with respect to R the commutative exact diagram:

$$\begin{array}{ccccccccccccc}
 & & & & & & & & & & \operatorname{Ext}_R^{c-1}(K_{R/J}, R)(-r) & & \\
 & & & & & & & & & & \downarrow & & \\
0 \to & R & \to & F_1^* & \to \ldots \to & F_{c-1}^* & \to & E^* & \to & & \operatorname{Ext}_R^c(R/I, R) & & \to 0 \\
 & \| & & \downarrow & & \downarrow & & \downarrow & & & \downarrow \alpha & & \\
0 \to & R & \to & D_{c-1}(s) & \to \ldots \to & D_1(s) & \to & R(s) & \to & & \operatorname{Ext}_R^c(R/\mathfrak{c}, R) & & \to 0.
\end{array}$$

Since $K_{R/J}$ has dimension $n+1-c$ we obtain by duality $\operatorname{Ext}_R^{c-1}(K_{R/J}, R) = 0$. Moreover, we have already seen that $\operatorname{Ext}_R^c(R/I, R) \cong K_{R/I}(n+1)$ and $\operatorname{Ext}_R^c(R/\mathfrak{c}, R) \cong R/\mathfrak{c}(s)$. It follows that α is injective and, by comparison with the standard exact sequence, $\operatorname{coker} \alpha \cong R/J(s)$. Thus, the mapping cone procedure provides an exact sequence which begins with $R/J(s)$ and ends with R. However, it can be shown that this last module can be canceled. The result is the N-type resolution of J as claimed because E^* meets the cohomological requirements (cf. [3], Theorem 4.25 and [37], Theorem 3.8).

The claimed E-type resolution of J can be obtained by similar arguments. $\square$

Using Remark 7.5 we obtain as first consequence the generalization of Lemma 7.3.

Corollary 7.8. *Let $V, W \subset \mathbb{P}^n$ be equidimensional subschemes. If $W \in \mathcal{L}_V$ then there is an integer t such that*

$$H^i_*(\mathcal{I}_W) \cong H^i_*(\mathcal{I}_V)(t) \quad \textit{for all } \; i = 1, \ldots, \dim V.$$

This result gives necessary conditions for V and W being in the same even liaison class. In order to state a stronger consequence of Proposition 7.7 we need.

Definition 7.9. The E-type and the N-type resolution, respectively, of I are said to *minimal* if it is not possible to cancel free direct summands. They are uniquely determined by I up to isomorphism of complexes.

Let $\varphi(I)$ denote the last non-vanishing module in a minimal E-type resolution of I, and let $\psi(I)$ denote the second non-vanishing module in a minimal N-type resolution of I.

We consider φ and ψ as maps from the set of ideals into the set of maximal R-modules. If the E-type and N-type resolutions in Definition 7.4 are minimal then we have $\varphi(I) = E$ and $\psi(I) = N$. Basically we will ignore possible free direct summands of $\varphi(I)$ and $\psi(I)$. This is formalized as follows.

Definition 7.10. Two graded maximal R-modules M and N are said to be *stably equivalent* if there are finitely generated, free R-modules F, G and an integer t such that

$$M \oplus F \cong N(t) \oplus G.$$

The *stable equivalence class* of M is the set

$$[M] := \{N \mid N \text{ is stably equivalent to } M\}.$$

Using Proposition 7.7 repeatedly we get the following relation between even liaison and certain stable equivalence classes.

Theorem 7.11 (Rao's correspondence)**.** *The map φ induces a well-defined map $\Phi : \mathcal{M}_c \to \mathcal{M}_E, \mathcal{L}_I \mapsto [\varphi(I)]$, from the set $\mathcal{M}_c$ of even liaison classes of unmixed ideals in R of codimension c into the set $\mathcal{M}_E$ of stable equivalence classes of finitely generated $(c+1)$-syzygies.*

The map ψ induces a well-defined map $\Psi : \mathcal{M}_c \to \mathcal{M}_N, \mathcal{L}_I \mapsto [\psi(I)]$, from the set $\mathcal{M}_c$ of even liaison classes into the set $\mathcal{M}_N$ of stable equivalence classes of finitely generated reflexive modules N which satisfy $H^i_{\mathfrak{m}}(N) = 0$ for all i with $n - c + 2 \leq i \leq n$.

Remark 7.12. (i) Rao's correspondence provides the following diagram with two commuting squares

$$\begin{array}{ccc}
\mathcal{M}_c & \xrightarrow{\Phi} & \mathcal{M}_E \\
\downarrow \alpha & & \downarrow \beta \\
\mathcal{M}_c & \xrightarrow{\Psi} & \mathcal{M}_N \\
\downarrow \alpha & & \downarrow \beta \\
\mathcal{M}_c & \xrightarrow{\Phi} & \mathcal{M}_E
\end{array}$$

where α is induced by direct linkage and β is induced by dualization with respect to R.

(ii) Combining Rao's correspondence with Horrocks' classification of stable equivalence classes of vector bundles on $\mathbb{P}^n$ in terms of cohomology groups and extensions [55] gives a stronger result than Corollary 7.8 in the case of locally Cohen-Macaulay subschemes. Horrocks' result gives for example: if C is a curve then the stable equivalence class of $\psi(I_C)$ is determined by $H^1_*(\mathcal{I}_C) \cong H^2_{\mathfrak{m}}(\psi(I_C))$. For a locally Cohen-Macaulay surface $S \subset \mathbb{P}^n$ the stable equivalence class of $\psi(I_S)$ is determined by the triple $(H^2_{\mathfrak{m}}(\psi(I_S)), H^3_{\mathfrak{m}}(\psi(I_S)), \eta)$ where $\eta \in \operatorname{Ext}^2_R(H^3_{\mathfrak{m}}(\psi(I_S)), H^2_{\mathfrak{m}}(\psi(I_S)))$. For the modules associated to schemes of dimension ≥ 3 Horrocks' classification becomes less elegant.

In particular, the modules $H^1_*(\mathcal{I}_S)$ and $H^2_*(\mathcal{I}_S)$ are not enough to determine the even liaison class. This is illustrated, for instance, in Example 7.14.

The next result gives more information on Rao's correspondence.

Proposition 7.13. *For every $c \geq 2$ the maps Φ and Ψ occurring in Rao's correspondence are surjective.*

Proof. Fixing c it suffices to show the claim for one of the maps according to the previous remark.

Let $c = 2$. Let $M \in \mathcal{M}_N$ be a module of rank r. Then for $s \gg 0$ a sufficiently general map $R^{r-1}(-s) \to M$ provides an exact sequence

$$0 \to R^{r-1}(-s) \to M \to I(t) \to 0$$

where t is an integer and I is an unmixed ideal of codimension two. This result is sometimes referred to as Theorem of Bourbaki.

For $c \geq 3$ the claim is shown in [2]. $\square$

Rao's correspondence gives the strongest known necessary conditions for two subschemes belonging to the same even G-liaison class. For even CI-liaison classes of ideals of codimension $c \geq 3$ there are additional necessary conditions (cf. [57], [61]).

The next example illustrates the fact that Rao's correspondence provides stronger necessary conditions than Corollary 7.8.

Example 7.14. The Koszul complex resolves the ideal $\mathfrak{m} = (x_0, \ldots, x_4)$ over $R := K[x_0, \ldots, x_4]$

$$\begin{tikzcd}
0 \arrow[r] & R(-5) \arrow[r] & R^5(-4) \arrow[rr] \arrow[dr] & & R^{10}(-3) \arrow[r] & R^{10}(-2) \arrow[rr] \arrow[dr] & & R^5(-1) \arrow[r] & \mathfrak{m} \arrow[r] & 0. \\
& & & \Omega^3 \arrow[ur] \arrow[dr] & & & \Omega^1 \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

The modules Ω^3 and Ω^1 are defined as the indicated syzygy modules.

There is a surface $S \subset \mathbb{P}^4$ admitting an exact sequence

$$0 \to (\Omega^3(-1))^2 \xrightarrow{\alpha} R(-4) \oplus (\Omega^1(-3))^2 \to I_S \to 0.$$

If the map α is general enough then S is a smooth rational surface of degree 10 ([32], Example B1.15). Moreover, its deficiency modules are

$$\begin{array}{rcl} H^1_*(\mathcal{I}_S) & \cong & K^2(-3) \\ H^2_*(\mathcal{I}_S) & \cong & K^2(-1). \end{array}$$

Denote by Ω^2 the second syzygy module of $\mathfrak{m}$ in the Koszul complex above. There is a surface $T \subset \mathbb{P}^4$ such that there is an exact sequence

$$0 \to R^{15}(-4) \to (\Omega^2)^2 \oplus (\Omega^1(-2))^2 \to I_T \to 0.$$

The deficiency modules of T are

$$\begin{array}{rcl} H^1_*(\mathcal{I}_T) & \cong & K^2(-2) \\ H^2_*(\mathcal{I}_T) & \cong & K^2. \end{array}$$

Thus, we obtain

$$H^i_*(\mathcal{I}_S) \quad \cong \quad H^i_*(\mathcal{I}_T)(-1) \quad \text{for} \quad i = 1, 2.$$

This leaves open the possibility that S and T are evenly linked. But in fact, S and T belong to different even liaison classes because $\varphi(I_S)$ and $\varphi(I_T)$ are not stably equivalent ([87], Example 7.4). Taking concrete examples for S and T this can be checked by looking at their hyperplane sections. Let $H \subset \mathbb{P}^4$ be a general hyperplane. It is not difficult to see that $S \in \mathcal{L}_T$ would imply $S \cap H \in \mathcal{L}_{T \cap H}$, but the latter is impossible because

$$\mathfrak{m} \cdot H^1_*(\mathcal{I}_{T \cap H}) = 0, \quad \text{but} \quad \mathfrak{m} \cdot H^1_*(\mathcal{I}_{S \cap H}) \neq 0.$$

In other words, the surface T is arithmetically Buchsbaum, but the surface S is not arithmetically Buchsbaum. Using Rao's correspondence one can show that the property of being arithmetically Buchsbaum is preserved under direct linkage (cf. [99]. [87]) which again implies $S \notin \mathcal{L}_T$.

8. Sufficient conditions for being linked

In Section 7 we have seen that Rao's correspondence relates even liaison classes to certain stable equivalence classes. Moreover, this correspondence is surjective. Thus, an ideal result would be an affirmative answer to the following.

Main question 8.1. Are the maps Φ and Ψ in Rao's correspondence injective for all $c \geq 2$?

In this section we will discuss this question. It is worthwhile to point out special cases of the Main question. We begin with a definition.

Definition 8.2. A subscheme $V \subset \mathbb{P}^n$ is *licci* if it is in the CI-liaison class of a complete intersection. V is *glicci* if it is in the G-liaison class of a complete intersection.

Remark 8.3. (i) Since for an arithmetically Cohen-Macaulay subscheme V the modules $\Phi(I_V)$ and $\Psi(I_V)$ are free, i.e. stably equivalent to the zero module, in this case the Main question takes the form:

Question 1: Is it true that a subscheme V is arithmetically Cohen-Macaulay if and only if it is glicci, i.e. in the G-liaison class of a complete intersection?

(ii) Let $C, D \subset \mathbb{P}^n$ be two curves. Then it is not to difficult to see, and it is a special case of Horrocks' results [55], that the following conditions are equivalent:

(a) $\Phi(I_C)$ and $\Phi(I_D)$ are stably equivalent.
(b) $\Psi(I_C)$ and $\Psi(I_D)$ are stably equivalent.
(c) $H^1_*(\mathcal{I}_C) \cong H^1_*(\mathcal{I}_D)(t)$ for some $t \in \mathbb{Z}$.

Hence, for curves the main question specializes to:

Question 2: Is it true that two curves $C, D \subset \mathbb{P}^n$ belong to the same even G-liaison class if and only if $H^1_*(\mathcal{I}_C) \cong H^1_*(\mathcal{I}_D)(t)$ for some $t \in \mathbb{Z}$?

(iii) Strictly speaking we should ask the Main question and the two questions above for G-liaison and CI-liaison, separately. We state only the Main question for CI-liaison.

Question 3: Let $V, W \subset \mathbb{P}^n$ be two equidimensional subschemes of the same codimension. Is it true that V and W belong to the same even CI-liaison class if and only if $\Phi(I_V)$ and $\Phi(I_W)$ are stably equivalent?

For subschemes of codimension two the answer to all these questions is 'yes' because of the following result which is essentially due to Rao [94] (cf. also [90], [85]).

Theorem 8.4. *Let $I, I' \subset R$ be unmixed homogeneous ideals of codimension 2 with N-type resolutions*

$$0 \to \bigoplus_{i=1}^{s} R(-a_i) \xrightarrow{\delta} N \to I \to 0$$

and

$$0 \to \bigoplus_{i=1}^{s} R(-b_i) \xrightarrow{\varepsilon} N(h) \to I' \to 0.$$

Then I and I' belong to the same even liaison class.

Proof. We only outline the proof but give enough details to see where the problems are in extending the argument for ideals of higher codimension.

Case 1: Suppose N is a free R-module. Then I and I' are standard determinantal ideals and the claim follows from the more general Theorem 8.7.

Case 2: Suppose that N is not free. Then, possibly after linking I and I' in an even number of steps to new ideals, we may assume that N does not have a free direct summand.

Write $\operatorname{im}\delta = (m_1, \ldots, m_s)$ and $\operatorname{im}\varepsilon = (n_1, \ldots, n_s)$ where $m_iR = \delta(R(-a_i))$ and $n_iR = \varepsilon(R(-b_i))$. Suppose that $m_i = n_i$ for $i < t \leq s$.

We want to show that we can find ideals $I_1 \in \mathcal{L}_I$ and $I'_1 \in \mathcal{L}_{I'}$ having N-type resolutions where $m_i = n_i$ for $i \leq t$. Then, repeating this process at most s times our statement follows.

Choose an integer $p \gg 0$ and elements $u, v \in [N]_p$ whose images f_1, f_2 in I and g_1, g_2 in I' generate complete intersections $\mathfrak{c}$ and $\mathfrak{c}'$, respectively. Put $J = \mathfrak{c} : I$ and $J' = \mathfrak{c}' : I'$. According to Proposition 7.7 these ideals have E-type resolutions as follows:

$$0 \to N^*(-2p) \to R^2(-p) \oplus \bigoplus_{i=1}^{s} R(a_i - 2p) \to J \to 0,$$

$$0 \to N^*(2h - 2p) \to R^2(h - p) \oplus \bigoplus_{i=1}^{s} R(b_i + 2h - 2p) \to J' \to 0.$$

Let $f \in J$ be the generator of the image of $R(a_t - 2p)$ in J and let $g \in J'$ be the generator of the image of $R(b_t + 2h - 2p)$; f and g are not zero because N does not have a free direct summand. Since $\{f_1, f_2\}$ and $\{g_1, g_2\}$ are regular sequences it is possible to find $\lambda, \mu \in K$ such that $\mathfrak{d} = (f, f')$ and $\mathfrak{d}' = (g, g')$ are complete intersections where $f' = \lambda f_1 + \mu f_2$ and $g' = \lambda g_1 + \mu g_2$. Put $I_1 = \mathfrak{d} : J$ and $I_1' = \mathfrak{d}' : J'$. Since N does not have a free direct summand the E-type resolutions of J and J' above must be minimal. It follows that f, f' are minimal generators of J and that g, g' are minimal generators of J'. Therefore we can split off $R(-p) \oplus R(a_t - 2p)$ respectively $R(h-p) \oplus R(b_t + 2h - 2p)$ in the N-type resolution of I_1 respectively I_1' given by Proposition 7.7. The resulting resolutions are:

$$0 \to \bigoplus_{i \neq t} R(a_t - a_i - p) \oplus R(a_t - 2p) \xrightarrow{\alpha} N(a_t - p) \to I_1 \to 0,$$

$$0 \to \bigoplus_{i \neq t} R(b_t - b_i + h - p) \oplus R(b_t + 2h - 2p) \xrightarrow{\alpha'} N(b_t + h - p) \to I_1' \to 0$$

where the image of α is generated by $m_1, \dots, m_{t-1}, m_{t+1}, \dots, m_s, \lambda u + \mu v$ and the image of α' is generated by $n_1, \dots, n_{t-1}, n_{t+1}, \dots, n_s, \lambda u + \mu v$. This means that we have reached our goal by replacing m_t and n_t, respectively, by $\lambda u + \mu v$. $\square$

Let us look at this proof in case the codimension of I and I' is at least three. We can still split off terms in the N-type resolutions of $I_1$and I_1' at the end of these resolutions (as in the proof above). But since the resolutions are longer this is not enough to guarantee the splitting at the beginning of the resolution which would be needed to complete the argument.

As pointed out in Remark 8.3, the last result has an implication for space curves.

Corollary 8.5. *Let $C, D \subset \mathbb{P}^3$ be two curves. Then $D \in \mathcal{L}_C$ if and only if $H^1_*(\mathcal{I}_C) \cong H^1_*(\mathcal{I}_D)(t)$ for some $t \in \mathbb{Z}$.*

We still have to prove Case 1 in the previous proposition, which is a result of Gaeta. It can be generalized to arbitrary codimension [61]. For this we recall the following.

Definition 8.6. If A is a homogeneous matrix, we denote by $I(A)$ the ideal of maximal minors of A. If $\varphi : F \to G$ is a homomorphism of free graded R-modules then we define $I(\varphi) = I(A)$ for any homogeneous matrix A representing φ after a choice of basis for F and G. A codimension $c+1$ ideal $I \subset R$ will be called a *standard determinantal ideal* if $I_X = I(A)$ for some homogeneous $t \times (t+c)$ matrix, A. In a similar way we define standard determinantal subscheme of $\mathbb{P}^n$.

It is well-known that every standard determinantal subscheme is arithmetically Cohen-Macaulay.

Now we can state one of the main results of [61]. The case of codimension two was due to Gaeta, and this generalization thus bears his name.

Theorem 8.7 (Generalized Gaeta theorem)**.** *Every standard determinantal ideal is glicci.*

Proof. The proof is essentially an algorithm describing how the required links can be achieved. We outline the steps of this algorithm but refer to [61], Theorem 3.6 for the complete proof. The interested reader is invited to run the algorithm with a concrete example.

Let $I \subset R$ be a standard determinantal ideal of codimension $c+1$. Thus, there is a homogeneous $t \times (t+c)$ matrix A with entries in R such that $I = I(A)$. If $t = 1$ then I is a complete intersection and there is nothing to prove. Let $t > 1$. Then our assertion follows by induction on t if we have shown that I is evenly G-linked to a standard determinantal scheme I' generated by the maximal minors of a $(t-1) \times (t+c-1)$ matrix A'. Actually, we will see that A' can be chosen as the matrix which we get after deleting an appropriate row and column of the matrix A and that then I and I' are directly G-linked in two steps. In order to do that we proceed in several steps.

Step I: Let B be the matrix consisting of the first $t+c-1$ columns of A. Then the ideal $\mathfrak{a} := I(B)$ has codimension c, i.e. it is a standard determinantal ideal.

Possibly after elementary row operations we may assume that the maximal minors of the matrix A' consisting of the first $t-1$ rows of B generate an ideal of (maximal) codimension $c+1$. Denote this standard determinantal ideal by $I' := I(A')$.

Step II: Possibly after elementary column operations we may assume that the maximal minors of the matrix A_1 consisting of the first $t-1$ columns of A generate an ideal of (maximal) codimension two. Put $J = I(A_1)$. Let d be the determinant of the matrix which consists of the first $(t-1)$ and the last column of A. Then one can show that

(i) $\mathfrak{a} : d = \mathfrak{a}$.
(ii) $I = (\mathfrak{a} + dR) : J$.
(iii) $\mathfrak{a} + dJ^{c-1}$ is a Gorenstein ideal of codimension $c+1$.
(iv) $\deg \mathfrak{a} + dJ^{c-1} = \deg d \cdot \deg \mathfrak{a} + \deg(\mathfrak{a} + J^{c-1})$.

Step III: Consider for $i = 0, \ldots, c$ the ideals $I_S + J^i$. These are Cohen-Macaulay ideals of degree

$$\deg(\mathfrak{a} + J^i) = i \cdot [\deg d \cdot \deg \mathfrak{a} - \deg I].$$

The proof involves in particular a deformation argument.

Step IV: Comparing degrees it is now not to difficult to check that

$$(\mathfrak{a} + dJ^{c-1}) : I = \mathfrak{a} + J^c.$$

Step V: Let d' be the determinant of the matrix which consists of the first $(t-1)$ columns of A'. Then, similarly as above, $\mathfrak{a} + d'J^{c-1}$ is a Gorenstein ideal of codimension $c+1$ and

$$(\mathfrak{a} + d'J^{c-1}) : I' = \mathfrak{a} + J^c.$$

Step VI: Step V says that the ideal I' is directly G-linked to $\mathfrak{a} + J^c$ while Step IV gives that I is directly G-linked to $\mathfrak{a} + J^c$. Hence the proof is complete. ☐

Example 8.8. Let $C \subset \mathbb{P}^n$ denote a rational normal curve. It is well-known that after a change of coordinates we may assume that the homogeneous ideal of C is generated by the maximal minors of the matrix

$$\begin{pmatrix} x_0 & \dots & x_{n-1} \\ x_1 & \dots & x_n \end{pmatrix}.$$

Hence, C is standard determinantal and therefore glicci by Gaeta's theorem.

On the other hand the curve C has a linear free resolution, i.e. its minimal free resolution has the shape

$$0 \to R^{\beta_{n-1}}(-n) \to \dots \to R^{\beta_1}(-2) \to I_C \to 0.$$

Hence, [57], Corollary 5.13 implies for $n \geq 4$ that the curve C is not licci, i.e. not in the CI-liaison class of a complete intersection.

Let us look back to the questions posed at the beginning of this section. The previous example shows that the answer to Question 3 is 'no', i.e. Rao's correspondence is not injective for CI-liaison in codimension ≥ 3. Gaeta's theorem indicates that the situation might be different for G-liaison. In fact, there is more evidence that Question 1 could have an affirmative answer. To this end we consider certain monomial ideals.

Definition 8.9. A monomial ideal $J \subset R$ is said to be *stable* if

$$m = x_0^{a_0} \cdots x_n^{a_n} \in J \text{ and } a_i > 0 \text{ imply } \frac{x_j}{x_i} \cdot m \in J$$

for all $1 \leq j < i \leq n$.

Theorem 8.10. *Suppose that the ground field K is infinite. Then every Cohen-Macaulay Borel-fixed monomial ideal is glicci.*

Proof. The main tools are basic double links and liftings of monomial ideals (cf. [75]). We only outline the main steps of the proof and refer for details to [76]. Moreover, we assume for simplicity that K has characteristic zero.

Step I: Let $J \subset R$ be a Cohen-Macaulay stable monomial ideal of codimension $c+1$. Denote by

$$\alpha := \min\{t \in \mathbb{Z} \mid [I]_t \neq 0\}$$

its initial degree. Then there are uniquely determined Artinian stable ideals $I_0, \dots, I_\alpha \subset T := K[x_1, \dots, x_c]$ such that

$$I_0 \subset I_1 \subset \dots \subset I_\alpha = T$$

and

$$\begin{aligned} J &= I_0R + x_0I_1R + x_0^2I_2R + \dots + x_0^\alpha I_\alpha R \\ &= I_0R + x_0I' \end{aligned}$$

where $I' = I_1R + x_0I_2R + \dots + x_0^{\alpha-1}I_\alpha R$.

It follows that

(i) I_0R is a Cohen-Macaulay ideal of codimension c.
(ii) $I_0R \subset I'$ because $I_0R \subset I_1R \subset I'$.
(iii) I' is a Cohen-Macaulay ideal of codimension $c+1$.

Step II: Now we want to lift monomial ideals in T to reduced ideals in $S := T[x_0]$.

Consider the lifting map $\lambda : \{\text{monomials in } T\} \to \{\text{monomials in } S\}$ given by

$$\prod_{j=1}^{c} x_j^{a_j} \mapsto \prod_{j=1}^{c} \left(\prod_{i=0}^{a_j-1} (x_j + i x_0) \right).$$

(Here the assumption on the characteristic is used. In general, one just has to choose sufficiently general linear forms in order to replace monomials by products of linear forms as above.) For example, we get $\lambda(x_1^3 x_2^2) = x_1(x_1+x_0)(x_1+2x_0)x_2(x_2+x_0)$.

The properties of the lifting map ensure that $\lambda(I_0)$ is a reduced ideal defining a set of points in $\mathbb{P}^c$. Therefore this set has the property G_1.

Step III: Using the stability of J one can show that $\lambda(I_0)R \subset I'$ and

$$J = \lambda(I_0)R + x_0 I'.$$

Thus, J is a basic double link of I' and Proposition 6.3 shows that $I' \in \mathcal{L}_J$. But the initial degree of I' is $\alpha - 1$. Repeating this argument successively we see that $I_0R + x_0R \in \mathcal{L}_J$. Hence it is sufficient to show that $I_0R + x_0R$ is glicci. But this follows because I_0R is glicci by induction on the codimension. The claim is clearly true for ideals of codimension one. □

Theorem 8.10 is of a more general nature than it is apparent from its formulation.

Remark 8.11. Let $V \subset \mathbb{P}^n$ be an arithmetically Cohen-Macaulay subscheme. It is well-known its generic initial ideal $gin(I_V)$ is a stable ideal and defines an arithmetically Cohen-Macaulay subscheme which is a deformation of the original scheme V. Indeed, the fact that $gin(I_V)$ is stable is due to Galligo [39]; that it gives a flat deformation is due to Bayer [6]; that it is again Cohen-Macaulay follows from a result of Bayer and Stillman (cf. [35], Theorem 15.13). Thus our result says that every arithmetically Cohen-Macaulay subscheme admits a flat deformation which is glicci. In other words, we have found an affirmative answer to Question 1 "up to flat deformation."

In view of Remark 8.11, we consider Theorem 8.10 as the strongest evidence that Question 1 might have an affirmative answer. However, there is also other evidence.

Remark 8.12. The results about linear systems (Theorem 6.1) can be used to show that many arithmetically Cohen-Macaulay subschemes are glicci. This becomes particularly effective for divisors on arithmetically Cohen-Macaulay subschemes with known Picard group. Some typical results of this approach are

(i) All arithmetically Cohen-Macaulay curves on a general smooth rational arithmetically Cohen-Macaulay surface in $\mathbb{P}^4$ are glicci ([61], Corollary 8.9).
(ii) Let $S \subset \mathbb{P}^4$ be a general arithmetically Cohen-Macaulay surface such that all the entries of its Hilbert-Burch matrix have positive degree. Then all arithmetically Cohen-Macaulay curves on S are glicci ([26]).
(iii) Effective arithmetically Cohen-Macaulay divisors on a smooth rational normal scroll are glicci ([24]).
(iv) Every general set of points in $\mathbb{P}^3$ on a nonsingular quadric surface is glicci ([53]). More generally, every general set of points on a smooth rational surface scroll is glicci ([24], Theorem 3.4.2).

One of the few sufficient conditions for linkage in higher codimension was mentioned in Remark 2.13, and now we sketch the proof.

Proposition 8.13. *Any two complete intersections of the same codimension are CI-linked.*

Proof. (Sketch of proof from [101])
The proof rests on the following observation: If $I_{X_1} = (F_1, \ldots, F_{c-1}, F)$ and $I_{X_2} = (F_1, \ldots, F_{c-1}, G)$ are two complete intersections of codimension c then they are directly linked by the complete intersection $I_X = (F_1, \ldots, F_{c-1}, FG)$. Then the proof follows by changing one entry at a time. □

From this it would follow that if one could show that every arithmetically Gorenstein scheme is glicci, then all arithmetically Gorenstein schemes are in the same G-liaison class. However, this is not known. It is true if the codimension is at most three. Then an arithmetically Gorenstein subscheme is even licci ([107]).

Moreover, Hartshorne [53] has proposed interesting examples. He suspects that a set of 20 general points in $\mathbb{P}^3$ as well as the general curve in the irreducible component of the Hilbert scheme of curves in $\mathbb{P}^4$ of degree 20 and genus 26 containing standard determinantal curves is not glicci.

There are also some results for non-arithmetically Cohen-Macaulay subschemes indicating that even the Main question might have an affirmative answer:

- Hartshorne [53] and Lesperance [64] independently showed that any two sets of two skew lines in $\mathbb{P}^4$ are G-linked. (See also Conjecture 10.4.) Hartshorne also obtained partial results on other curves with Rao module k.
- Lesperance [64] showed that curves in $\mathbb{P}^4$ consisting of unions of two plane curves are (at least "usually") linked if and only if they have the same Rao module.
- Lesperance [65] showed that if C and C' are degenerate arithmetically Buchsbaum curves in $\mathbb{P}^4$ (not necessarily in the same hyperplane) then C and C' are evenly G-linked if and only if they have isomorphic Rao modules up to shift.
- Casanellas and Miró-Roig [25], [26] showed the same for many subschemes of small degree (not necessarily curves), especially unions of linear varieties; their idea was to view them as divisors on a suitable rational normal scroll.
- Nagel, Notari and Spreafico [89] proved for double lines in $\mathbb{P}^n$ and for some other non-reduced curves on lines, that they are evenly linked if and only if they have isomorphic Rao modules up to shift.

The proof of the last result differs from the others by not using the result about the G-liaison classes of divisors on arithmetically Cohen-Macaulay subschemes with the property G_1. Indeed, the non-reduced curves that are considered are not even divisors on a generically Gorenstein surface.

Note also that the Hartshorne-Rao modules of the curves considered in the first results mentioned above are rather simple while the curves studied in [88], [89] can have a rather complicated Hartshorne-Rao module.

9. The Structure of an Even Liaison Class

We have seen a rather complete description of when two subschemes are linked in codimension two. The main result is Theorem 8.4, and it is one of the main results of liaison theory. We have discussed to some extent the possibility of extending this result to higher codimension (e.g. Theorem 8.7), and we will continue to discuss it below. As we saw, it is more natural to consider *even* liaison.

Another natural question is whether the even liaison classes possess a common structure of any sort. We will see that in codimension two there is a nice answer. Again, one can try to extend it to higher codimension, and we will also discuss the evidence for and against this idea. The following remark sets up the background.

Remark 9.1. Let $V \subset \mathbb{P}^n$ be an equidimensional closed subscheme of codimension c. Let $M_i = H^i_*(\mathcal{I}_V)$ for $1 \leq i \leq \dim V = n - c$. Let $\mathcal{L}_V$ be the even liaison class of V. Note that

- The vector of graded modules $M_\bullet = (M_1, \ldots, M_{n-c})$ is an invariant of $\mathcal{L}_V$, up to shift (Lemma 7.3).
- There is a minimal shift of this vector that can occur among subschemes of $\mathbb{P}^n$ (Proposition 3.26; see Definition 3.27 for the definition of minimal shift).
- Hence there is a minimal shift of this vector among elements of $\mathcal{L}_V$ (which is not necessarily the same as the minimal shift among all subschemes in $\mathbb{P}^n$ with vector $\mathcal{L}_V$, except for curves in $\mathbb{P}^3$).
- Although leftward shifts of $M_\bullet$ may not exist, any rightward shift of $M_\bullet$ does exist thanks to Basic Double Linkage (Lemma 3.25, Remark 6.8).

Definition 9.2. If an element $W \in \mathcal{L}_V$ has cohomology which achieves the minimal shift, among elements of $\mathcal{L}_V$, guaranteed by Proposition 3.26, we say that W is a *minimal element of its even liaison class*, or that W *is in the minimal shift of* $\mathcal{L}_V$. We write $W \in \mathcal{L}^0_V$.

Example 9.3. Let Z_1 be the disjoint union in $\mathbb{P}^3$ of a line, λ, and a conic, Y. We have the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & I_{Z_1} \arrow[r] & I_\lambda \oplus I_Y \arrow[rr] \arrow[dr] & & R \arrow[r] & H^1_*(\mathcal{I}_{Z_1}) \arrow[r] & 0 \\
& & & I_\lambda + I_Y \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Since $I_\lambda + I_Y$ contains three independent linear forms, we conclude that $H^1_*(\mathcal{I}_{Z_1}) \cong K[x]/(x^2)$ for some linear form x. Therefore the module is one-dimensional in each of degrees 0 and 1, and zero everywhere else. However, notice that the module structure is not trivial: multiplication from the degree 0 component to the degree 1 component by the linear form x is not zero.

Note further that this curve is in the minimal shift of its even liaison class, thanks to the bound (3.2) which says that in negative degree the dimensions have to be strictly increasing.

Now consider a Buchsbaum curve Z_2, obtained via Liaison Addition as in Example 6.10, with deficiency module which is 1-dimensional in each of two consecutive degrees. The smallest such curve that can be so constructed is obtained by choosing C_1 and C_2 in Example 6.10 to each be a pair of skew lines, and $\deg F_1 = 2$, $\deg F_2 = 3$. Then the first non-zero component of $H^1_*(\mathcal{I}_{Z_1})$ occurs in degree 2 which, thanks to Proposition 6.11, is the minimal shift.

Note that the *structure* of these two modules, $H^1_*(\mathcal{I}_{Z_1})$ and $H^1_*(\mathcal{I}_{Z_2})$, is different (the latter is annihilated by all linear forms), even though dimensionally they are the same. Hence they are not in the same (even) liaison class.

For an example of surfaces where even the modules are isomorphic but the liaison classes are different, see Example 7.14.

Remark 9.4. Let $V \subset \mathbb{P}^n$ be temporarily an equidimensional scheme of codimension $c \geq 2$. Then it is clear how to adapt the above definition of $\mathcal{L}^0_V$. Note, that we have already defined the (cohomological) minimal shift in Definition 3.27. Strictly speaking we should distinguish even a third notion of minimal shift suggested by Rao's correspondence. This provides the following list:

(i) The (cohomological) minimal shift of V is the integer

$$c(V) := \min \left\{ t \in \mathbb{Z} \mid \begin{array}{c} \text{There is a subscheme } W \subset \mathbb{P}^n \text{ of codimension } c \text{ with} \\ H^i_*(\mathcal{I}_W) \cong H^i_*(\mathcal{I}_V)(-t) \end{array} \right\}.$$

(ii) The minimal Rao shift of V is the integer

$$r(V) := \min \left\{ t \in \mathbb{Z} \mid \begin{array}{c} \text{There is a subscheme } W \subset \mathbb{P}^n \text{ of codimension } c \text{ with} \\ \varphi(I_V) \oplus F \cong \varphi(I_W)(-t) \oplus G \text{ for free } R\text{-modules } F, G \end{array} \right\}.$$

(iii) The minimal shift of the even G-liaison class $\mathcal{L}_V$ is the integer

$$l(V) := \min \left\{ t \in \mathbb{Z} \mid \begin{array}{c} \text{There is a subscheme } W \in \mathcal{L}_V \text{ with} \\ \varphi(I_V) \oplus F \cong \varphi(I_W)(-t) \oplus G \text{ for free } R\text{-modules } F, G \end{array} \right\}.$$

According to Remark 7.5 and Rao's correspondence we have the following inequalities

$$c(V) \leq r(V) \leq l(V).$$

Moreover, if V is a curve then $c(V) = r(V)$, but if the dimension of V is at least 2 we can have $c(V) < r(V)$.

If the codimension of V is two we get $r(V) = l(V)$ due to Theorem 8.4. It would be interesting to know if this equality is also true in codimension $c \geq 3$. This would follow from an affirmative answer to the Main question 8.1, but it is conceivable that the Main question has a negative answer and $r(V) = l(V)$ is still always true.

In [85], Proposition 5.1 a lower bound for $r(V)$ is given which cannot be improved in general. It would be interesting to have a priori estimates for $c(V)$ and $l(V)$ as well.

We now describe a structure of an even liaison class, generally called the *Lazarsfeld-Rao property*. As remarked above, this property is only known to hold in codimension two, so we now make this assumption. Later we will discuss the possibility of extending it.

Let $\mathcal{L}$ be an even liaison class of codimension two subschemes of $\mathbb{P}^n$. For simplicity we will assume that the elements of $\mathcal{L}$ are locally Cohen-Macaulay, and of course they must be equidimensional. (The locally Cohen-Macaulay assumption was removed by Nagel [85] and by Nollet [90].)

As we have seen (e.g. Theorem 8.7), the arithmetically Cohen-Macaulay codimension two subschemes *form* an even liaison class. (In this case any two schemes are both evenly and oddly linked.) We thus assume that the elements of $\mathcal{L}$ are *not* arithmetically Cohen-Macaulay, so $M_\bullet$ is not zero (i.e. at least one of the modules, not necessarily all, is non-zero). Then it follows from Remark 9.1 that we can partition $\mathcal{L}$ according to the shift of $M_\bullet$:

$$\mathcal{L} = \mathcal{L}^0 \cup \mathcal{L}^1 \cup \mathcal{L}^2 \cup \cdots \cup \mathcal{L}^h \cup \ldots.$$

Here, $\mathcal{L}^0$ was defined in Definition 9.2 and consists of the minimal elements. Then $\mathcal{L}^h$ consists of those elements of $\mathcal{L}$ whose deficiency modules are shifted h degrees to the right of the minimal shift.

In Remark 6.5 we saw the notion of Basic Double CI-Linkage and in particular we gave the version for codimension two: Let V_1 be a codimension two subscheme of $\mathbb{P}^n$ and choose $F_2 \in I_{V_1}$ of degree d_2 and $F_1 \in R$ of degree d_1 such that (F_1, F_2) forms a regular sequence (i.e. a complete intersection). Then $F_1 \cdot I_{V_1} + (F_2)$ is the saturated ideal of a scheme Z which is CI-linked to V_1 in two steps. Furthermore,

$$H^i_*(\mathcal{I}_Z) \cong H^i_*(\mathcal{I}_{V_1})(-d_1) \qquad \text{for } i = 1, \ldots, n-2.$$

As sets, $Z = V_1 \cup V$ where V is the complete intersection defined by (F_1, F_2). Note that if $V_1 \in \mathcal{L}^h$ then $Z \in \mathcal{L}^{h+d_1}$. A concrete description of the two links can be given as follows (first noted in [63]): Let $A \in I_{V_1}$ be any homogeneous polynomial having no component in common with F_2. Then link V_1 to some intermediate scheme Y using the complete intersection (A, F_2), and link Y to Z using the complete intersection (AF_1, F_2).

One can also check, using various methods, that the E-type resolutions of V_1 and Z are related as follows. If $\mathcal{I}_{V_1}$ has an E-type resolution

$$0 \to \mathcal{E} \to \bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \mathcal{I}_{V_1} \to 0,$$

where $H^1_*(\mathcal{E}) = 0$, then $\mathcal{I}_Z$ has an E-type resolution

$$0 \to \mathcal{E}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^n}(-d_1 - d_2) \to \bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^n}(-d_1 - a_i) \oplus \mathcal{O}_{\mathbb{P}^n}(-d_2) \to \mathcal{I}_Z \to 0, \tag{9.1}$$

Note that the stable equivalence of $\mathcal{E}$ and $\mathcal{E}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^n}(-d_1 - d_2)$ is obvious.

The Lazarsfeld-Rao property says, basically, that in an even liaison class, all the minimal elements look alike and that the entire class can be built up from an arbitrary minimal element using Basic Double Linkage and deformation. More precisely, we have the following statement.

Theorem 9.5. (Lazarsfeld-Rao property) *Let $\mathcal{L}$ be an even liaison class of codimension two subschemes of $\mathbb{P}^n$.*

(a) *If $V_1, V_2 \in \mathcal{L}^0$ then there is a flat deformation from one to the other through subschemes all in $\mathcal{L}^0$.*

(b) *If $V_0 \in \mathcal{L}^0$ and $V \in \mathcal{L}^h$ $(h \geq 1)$ then there is a sequence of subschemes $V_0, V_1, \ldots, V_t$ such that for all i, $1 \leq i \leq t$, V_i is a basic double link of V_{i-1}, and V is a deformation of V_t through subschemes all in $\mathcal{L}^h$.*

We stress that the deformations mentioned in Theorem 9.5 are carried out entirely within the even liaison class $\mathcal{L}$. They preserve cohomology, not only dimensionally but even structurally.

Theorem 9.5 was first proved for codimension two locally Cohen-Macaulay subschemes of $\mathbb{P}^n$ in [4]. At approximately the same time, it was proved (as part of a much broader theory) for curves in $\mathbb{P}^3$ in [68]. It was proved for codimension two subschemes of a smooth arithmetically Gorenstein subscheme in [20]. Finally, in codimension two it was later extended to arbitrary unmixed ideals in [85] and [90]. We now give the general idea of the proof of [4], and refer the reader to that paper for the details, as well as to [68], [85] and [90].

Proof. (Sketch) There are three basic components of the proof.

1. (Bolondi, [16]) If $V_1, V_2 \in \mathcal{L}^h$ (in particular they have the same deficiency modules) and if they have the same Hilbert function then the desired deformation can be found. So it is reduced to a question of Hilbert functions.
2. If $V_1, V_2 \in \mathcal{L}^h$ and if they do *not* have the same Hilbert function then by studying locally free N-type resolutions one can show that there is a "smaller" V' in the even liaison class (i.e. $V' \in \mathcal{L}^{h'}$ for some $h' < h$). Combined with the first part, this proves that the minimal elements all lie in the same flat family.
3. Given $V_0 \in \mathcal{L}^0$ and $V \in \mathcal{L}^h$, by studying (9.1) and knowing that V_0 and V are linked in an even number of steps, it is possible to "predict" what basic double links are needed to start with V_0 and arrive at a scheme V_t with E-type resolution which agrees (except for the maps) with that of V, up to trivially adding free summands to both modules in the resolution. This means that V and V_t have the same Hilbert function and deficiency modules, so we again apply the first part.

□

Remark 9.6. (i) From the name "Lazarsfeld-Rao property" one would naturally expect that the paper [63] of Lazarsfeld and Rao was important in the development of the above theorem. In fact, it really inspired it (although many people doubted that something so general would hold). We can state the main result of [63] in the following way. For a curve $C \subset \mathbb{P}^3$, let

$$e(C) := \max\{t | h^2(\mathcal{I}_C(t)) \neq 0\} = \max\{t | h^1(\mathcal{O}_C(t)) \neq 0\} = \max\{t | h^0(\omega_C(-t)) \neq 0\}.$$

Then

a. If C lies on no surface of degree $e(C) + 3$ then $\mathcal{L}_C$ has the Lazarsfeld-Rao property and $C \in \mathcal{L}_C^0$.
b. If C lies on no surface of degree $e(C) + 4$ then furthermore C is the *only* element of $\mathcal{L}_C^0$.

For example, suppose that $C \subset \mathbb{P}^3$ is a set of ≥ 2 skew lines. Then $e(C) = -2$. Thus since C cannot lie on a surface of degree 1, part a. gives that $C \in \mathcal{L}_C^0$. If C furthermore does not lie on a quadric surface then C is the only minimal element of its even liaison class.

Similarly, one can apply it to rational curves, where $e(C) = -1$, and get analogous statements: a rational curve lying on a quadric surface is not minimal (it is linked to a set of skew lines), one lying on a cubic surface is minimal but not unique (it moves in a linear system) and one not lying on a cubic surface is the unique minimal curve. A different, more geometric approach to the minimality of skew lines and rational curves (not using [63]), and other related questions, can be found in [70].

(ii) Let us recall the concept of elementary CI-biliaison (in the case of curves). Let $C \subset \mathbb{P}^n$ be a curve which is an effective divisor on a complete intersection surface $S \subset \mathbb{P}^n$. Let F_1 be a hypersurface meeting S transversally such that $C \subset S \cap F_1$. Let C' be the curve linked to C by $S \cap F_1$. Choose a hypersurface F_2 such that it meets S transversally and $C' \subset S \cap F_2$. Denote by C'' the curve linked to C' by $S \cap F_2$. Then it is said that C'' is obtained from C by an *elementary CI-biliaison* on S. It is called *ascending* if $\deg F_2 - \deg F_1 \geq 0$, otherwise *decsending.* As already indicated in Remark 6.2 C'' is obtained from C by an elementary CI-biliaison on S if and only if $C'' \sim C + hH$. Observe that elementary CI-biliaison is a generalization of basic double CI-linkage (cf. Remark 6.5). Recently, in [104] R. Strano has obtained the following variant of the Lazarsfeld-Rao property: Let $C \subset \mathbb{P}^3$ be a curve which is not arithmetically Cohen-Macaulay. Then C can be obtained from a minimal curve in its even liaison class by finitely many ascending elementary CI-biliaisons. Thus, using the more general elementary biliaison instead of basic double links we can avoid the possible final deformation which is allowed in Theorem 9.5.

Remark 9.7. If one knows the Hilbert function of a curve C in $\mathbb{P}^3$ (or of a codimension two subscheme in general) then one can write the Hilbert function of all possible basic double links from C. Hence the Lazarsfeld-Rao property can be used to give a complete list of all possible $(d, g) =$ (degree,genus) combinations that occur in an even liaison class (g is the *arithmetic* genus), if one only knows it for a minimal element.

For example consider curves in $\mathbb{P}^3$ that are arithmetically Buchsbaum but not arithmetically Cohen-Macaulay. In Example 6.10 we saw one way to construct them, such that the result has its leftmost component in degree $2N - 2$ (where $N = \dim_k H^1_*(\mathcal{I}_C)$), which according to Proposition 6.11 makes it a minimal element of its even liaison class. Hence its degree, genus, and even its Hilbert function, are uniquely determined, thanks to the Lazarsfeld-Rao property. The following, from [72], is a complete list of the possible (d, g) that can occur for arithmetically Buchsbaum curves in $\mathbb{P}^3$ when $d \leq 10$. It includes two curves for which $N = 2$: one with $(d, g) = (8, 5)$ and $H^1_*(\mathcal{I}_C)$ concentrated in degree 2, and one with $(d, g) = (10, 10)$ and $\dim_k H^1_*(\mathcal{I}_C)_2 = \dim_k H^1_*(\mathcal{I}_C)_3 = 1$. The rest have $\dim_k H^1_*(\mathcal{I}_C) = 1$.

degree	2	3	4	5	6	7	8	9	10
genus	-1	d.n.e.	0	1	3	4,6	5,6,8 10	8, 9, 15	10, 11, 13, 15, 21

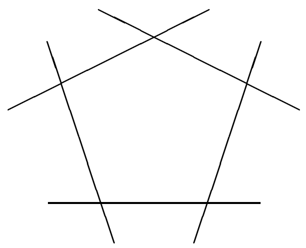
Note that there is no such curve of degree 3.

As was the case with the necessary and sufficient conditions for G-linkage, the biggest open problem is to find a way to extend these results to higher codimension. One intermediate situation was studied in [20] (cf. also [85]), where liaison was studied not in projective space but rather on a smooth arithmetically Gorenstein subvariety X of projective space. It was shown that codimension two liaison here behaves almost identically to that in $\mathbb{P}^n$, even though of course the objects being linked have codimension greater than two in $\mathbb{P}^n$. These results have been further generalized in [85] to codimension two subschemes of an arbitrary integral arithmetically Gorenstein subscheme.

One interesting difference concerns arithmetically Cohen-Macaulay subvarieties. Here we mean arithmetically Cohen-Macaulay in projective space (i.e. the deficiency modules vanish), but such a subvariety need not have a finite resolution over the Gorenstein coordinate ring R/I_X. It was shown that the notion of minimality still makes sense, viewed not in terms of the shift of the modules (which are zero) but rather in terms of N-type resolutions. Then it was shown that the Lazarsfeld-Rao property holds in such a situation on X.

Note that the linkage on X is by complete intersections on X, which however are only arithmetically Gorenstein as subschemes of $\mathbb{P}^n$. But if we turn to Gorenstein liaison in $\mathbb{P}^n$ with no such restriction, the situation becomes much less optimistic.

First, we can see right away that there is no hope for a statement which is identical to that for codimension two. The following example was taken from [72]. Consider the non-degenerate curve in $\mathbb{P}^4$ in the following configuration:



This curve is arithmetically Gorenstein. As such, it links two skew lines to a curve of degree 3 consisting of the disjoint union of a line and two lines meeting in a point. One checks that both of these curves have Rao module which is one dimensional, occurring in degree 0. Since this is the minimal shift, it is clear that the elements of $\mathcal{L}^0$ do not all have the same degree, hence are not in a flat family.

So if there is a nice structure for an even liaison class under Gorenstein liaison, what should the statement be? The next natural guess, due to Hartshorne [53] is that perhaps the elements of $\mathcal{L}^0$ satisfy the property that while there may be curves of different degrees, those curves of the same degree at least lie in a flat family. He showed this for the liaison class of two skew lines. However, it was shown to be false in general by Lesperance [64], who gave an example of two sets of curves "usually" in the same even liaison class which are in the minimal shift and have the same degree and even arithmetic genus, but which do not lie in the same flat family. His example was extended somewhat by Casanellas [24], who looked at the same kind of curves but in $\mathbb{P}^5$. (Lesperance was not able to show that all of his curves are in the same even liaison class, even though they do have the same Rao module. Casanellas showed that this obstacle disappears in $\mathbb{P}^5$.)

So at the moment no one has a good idea of how to find an analog to the Lazarsfeld-Rao property for Gorenstein liaison of subschemes of $\mathbb{P}^n$ of codimension ≥ 3. A first problem seems to be to find a good concept of a minimal element of an even G-liaison class. In the even liaison class $\mathcal{L}$ of a non-arithmetically Cohen-Macaulay subscheme of codimension two, the minimal elements are the elements of smallest degree in $\mathcal{L}$ and all these elements have the same Hilbert function. In particular, a non-arithmetically Cohen-Macaulay curve of degree two in $\mathbb{P}^3$ must be minimal in its even liaison class. In higher codimension the situation is very different. It is still true that two curves of degree two in $\mathbb{P}^n$ are in the same even liaison class if and only if their Hartshorne-Rao modules are isomorphic according to [88], but such curves can have different genera.

A naive idea would be to define the minimal elements in an even G-liaison class as the ones achieving the minimal shift and having minimal Hilbert polynomial. Consider the curves of degree two in $\mathbb{P}^n$ whose Hartshorne-Rao module is isomorphic to the ground field K. Such a curve can have every arithmetic genus g satisfying $-\frac{n-1}{2} \leq g \leq -1$, but it is non-degenerate if and only if $-\frac{n-1}{2} \leq g \leq 2-n$ ([88]). Thus, for $n \geq 4$ minimal curves in the sense just discussed were degenerate.

It should be remarked that the authors wonder if the Lazarsfeld-Rao property, even as it is stated in codimension two, might hold for CI-liaison in higher codimension. There are some encouraging result in [57].

10. Remarks on the different liaison concepts

We have already seen that for subschemes whose codimension is at least three, G-linkage and CI-linkage generate very different equivalence classes. In this section we want to discuss these differences a bit more systematically. Finally, we compare briefly the equivalence classes generated by (algebraic) CI-linkage and geometric CI-linkage.

As we have mentioned in Section 7, Rao's correspondence gives the only known method for distinguishing between G-liaison classes. The situation is different for CI-liaison. There are various invariants, numerical ([56]) as well as structural ([23], [57], [61]), which allow one to distinguish between CI-liaison classes of arithmetically Cohen-Macaulay subschemes. In order to give the flavour of such invariants, we state a particularly clean result which has been shown in [23] by algebraic means, whereas a more geometric proof has been given in [61], Proposition 6.8.

Theorem 10.1. *Suppose $V, W \subset \mathbb{P}^n$, $n \geq 4$, are arithmetically Cohen-Macaulay subschemes of codimension 3. If V and W belong to the same CI-liaison class then there are isomorphisms of graded R-modules*

$$H^i_{\mathfrak{m}}(K_V \otimes_R I_V) \cong H^i_{\mathfrak{m}}(K_W \otimes_R I_W) \quad \textit{for all } i = 1, \ldots, n-3.$$

In other words, the modules $H^i_{\mathfrak{m}}(K_V \otimes_R I_V)$ are invariants of the CI-liaison class of V. They must vanish if V is licci.

Corollary 10.2. *Let $V \subset \mathbb{P}^n$, $n \geq 4$, be an arithmetically Cohen-Macaulay subscheme of codimension 3. If V is licci then $H^i_{\mathfrak{m}}(K_V \otimes_R I_V) = 0$ for all $i = 1, \ldots, n-3$.*

Proof. Let $I \subset R$ be a complete intersection of codimension three. Then we have the following isomorphisms (ignoring degree shifts)

$$K_{R/I} \cong R/I;$$

thus

$$K_{R/I} \otimes_R I \cong I/I^2 \cong (R/I)^3.$$

Since $H^i_{\mathfrak{m}}(R/I) = 0$ for $i \leq n-3$ because R/I is Cohen-Macaulay, Theorem 10.1 proves the claim. □

For example, this result can used to reprove that the rational normal curve in $\mathbb{P}^4$ is not licci (cf. Example 8.8).

In [61], the previous theorem has been used to investigate CI-liaison classes of curves on a Castelnuovo surface.

Example 10.3. Let $S \subset \mathbb{P}^4$ be a general Castelnuovo surface, i.e. the blow-up of a set of 8 general points in $\mathbb{P}^3$ embedded into $\mathbb{P}^4$ by the linear system $\mid 4E_0 - 2E_1 - E_2 - \ldots - E_8 \mid$. Note that S is an arithmetically Cohen-Macaulay surface of degree 5 which contains a rational normal curve C of $\mathbb{P}^4$. Denote by H_S the general hyperplane section of S. Furthermore, denote by C_j any curve in the linear system $\mid C + jH_S \mid$. Then we have (cf. [61], Example 7.9)

(a) The curve C_j is not licci if $j \geq 0$.
(b) The curves C_i and C_j belong to different CI-liaison classes whenever $1 \leq i < j$ and $j \geq 3$.

Since we know that all arithmetically Cohen-Macaulay curves on S are glicci (cf. Remark 8.12) we obtain that the G-liaison class of C contains infinitely many CI-liaison classes.

So far CI-liaison invariants beyond the G-liaison invariants given by Rao's correspondence are known only for arithmetically Cohen-Macaulay subschemes. It seems plausible to expect such additional invariants also for non-arithmetically Cohen-Macaulay subschemes. The problem of finding them deserves further investigation. Here is possibly the simplest situation. In [71] the following conjecture was made.

Conjecture 10.4. *If C is a set of two skew lines in $\mathbb{P}^4$, spanning a hyperplane H, and if C' is another set of two skew lines in $\mathbb{P}^4$, spanning a hyperplane H', then C is in the CI-liaison class of C' if and only if $H = H'$.*

This conjecture would say that somehow the hyperplane H is a geometric invariant of the CI-liaison class of C, so there must be some other algebraic invariant in addition to the Rao module. We have seen above that Hartshorne and Lesperance independently showed that C and C' are in the same G-liaison class, so this invariant would not hold for G-liaison.

We have seen that liaison in codimension two has two natural generalizations in higher codimension: CI-liaison and G-liaison. The former can be understood as a theory about divisors on complete intersections while G-liaison is a theory about divisors on arithmetically Cohen-Macaulay schemes with property G_1. Thus, G-liaison is a much coarser equivalence relation than CI-liaison. It has the advantage that it is well suited for studying linear systems. The even CI-liaison classes are rather small. In fact, it seems very difficult to find enough invariants which would completely characterize an even CI-liaison class.

It is also worth mentioning two disadvantages of G-liaison. The first is related to our thin knowledge of arithmetically Gorenstein subschemes. Given a subscheme V, it is difficult to find "good" G-links of V; i.e. "good" arithmetically Gorenstein subschemes X containing V, where "good" often means small. For example, it is not too difficult to determine the smallest degree of a complete intersection containing V, while it is not known how to find an arithmetically Gorenstein subscheme of smallest degree containing V.

The second concerns lifting the information on hyperplane sections. If $V, W \subset \mathbb{P}^n$ are arithmetically Cohen-Macaulay subschemes and $H \subset \mathbb{P}^n$ is a general hyperplane such that $V \cap H$ and $W \cap H$ are linked by the complete intersection $\bar{X} \subset H$ then there is a complete intersection $X \subset \mathbb{P}^n$ linking V to W such that $\bar{X} = X \cap H$. The corresponding conclusion fails if we replace "complete intersection" by "arithmetically Gorenstein" (cf. [61], Example 2.12).

In Section 5 we defined geometric CI-linkage. It is also a symmetric relation, thus its transitive closure is an equivalence relation which is essentially the same as CI-liaison. However, we have to be a little bit careful what we mean here. If V is not a generic complete intersection then clearly it does not participate in a geometric CI-link. Thus, we make the following definition.

Definition 10.5. Let $H(c, n)$ denote the set of all equidimensional generic complete intersections of $\mathbb{P}^n$ of codimension c.

Note that this differs from the corresponding definition of Rao [94] not only in allowing arbitrary codimension, but also in removing his assumption that the schemes are locally Cohen-Macaulay.

Geometric CI-liaison is an equivalence relation on $H(c, n)$ while CI-liaison is an equivalence relation of the set of all equidimensional subschemes of $\mathbb{P}^n$ having codimension c. But if we restrict the latter to $H(c, n)$ we get the following:

Theorem 10.6. *Algebraic and geometric CI-linkage generate the same equivalence relation on $H(c, n)$. That is, if $V, W \in H(c, n)$ are two generic complete intersections such that there is a sequence of (algebraic) CI-links*

$$V \sim V_1 \sim \dots V_s \sim W$$

with all $V_i \in H(c, n)$ then there is a sequence of geometric CI-links from V to W.

For the proof we refer to [61], Theorem 4.14. The result generalizes Rao's Theorem 1.7 in [94] which deals with the case $c = 2$.

The last result leaves open the following problem. Suppose there are $V, W \in H(c, n)$ such that there is a sequence of algebraic CI-links

$$V \sim V_1 \sim \dots V_s \sim W$$

where some of the V_i are *not* generic complete intersections. Is there still a sequence of geometric CI-intersections from V to W?

The answer is known in codimension two. It uses the observation of Rao ([94], Remark 1.5) that for a given $f \in I_V$ where $V \in H(2, n)$ there is always a form $g \in I_V$ of sufficiently large degree such that the complete intersection defined by (f, g) links V to a scheme V' which is also a generic complete intersection. Combining this fact with an analysis of the arguments which establish injectivity of Rao's correspondence in codimension two one gets the following:

Theorem 10.7. *Let $V, W \in H(2, n)$ be two subschemes such that there is a sequence of (algebraic) CI-links*

$$V \sim V_1 \sim \dots V_s \sim W.$$

Then there is a sequence of geometric CI-links from V to W.

For details of the proof we refer to [61],Theorem 4.16.

It is an open question if the analogue of Theorem 10.7 is also true for subschemes of codimension $c \geq 3$.

11. Applications of Liaison

In this section we mention some applications of liaison that have been made in the literature. It is not at all intended to be a complete list.

11.1. **Construction of arithmetically Gorenstein schemes with nice properties.** Here we describe in somewhat more detail the result of [77] mentioned on page 22. It represents one of the few applications so far of Gorenstein liaison as opposed to complete intersection liaison.

It is an open question to determine what Hilbert functions are possible for Artinian Gorenstein graded K-algebras. Indeed, this seems to be intractable at the moment. However, it was shown by Harima [46] that the Hilbert functions of the Artinian Gorenstein graded K-algebras *with the Weak Lefschetz property* (cf. Definition 4.16) are precisely the SI-sequences (see page 22 for the definition). Another open question is to determine the possible Hilbert functions of reduced, arithmetically Gorenstein subschemes of $\mathbb{P}^n$ of any fixed codimension. Again, it is not clear if this problem can be solved or not, but in the same way as the Artinian case, we have a partial result. That is, in [77] it was shown that every SI-sequence gives rise to a reduced union of linear varieties which is arithmetically Gorenstein and whose general Artinian reduction has the Weak Lefschetz property.

Remark 11.1. It would be very nice to show that every reduced arithmetically Gorenstein subscheme has the property that its general Artinian reduction has the Weak Lefschetz property. If this were the case, then the result of [77] would give a classification of the Hilbert functions of reduced arithmetically Gorenstein subschemes of $\mathbb{P}^n$, namely they would be those functions whose appropriate difference is an SI-sequence.

The construction given in [77] is somewhat technical, and we give only the main ideas. One of the interesting points of this construction is that it works in completely the opposite direction from the usual application of liaison. That is, instead of starting with a scheme V and finding a suitable arithmetically Gorenstein scheme X containing it, we start with a (very reducible) arithmetically Gorenstein scheme X and find a suitable subscheme V to link using X. Here are the main steps of the proof.

(a) Suppose that we have a *geometric* link $V_1 \overset{X}{\sim} V_2$, where V_1 (and hence also V_2) are arithmetically Cohen-Macaulay, and X is arithmetically Gorenstein (not necessarily a complete intersection). Suppose you know the Hilbert function of V_1 and of X. Then using Corollary 5.11 we can write the Hilbert function of V_2 (see also Example 5.15 (iii)). From the exact sequence

$$0 \to I_X \to I_{V_1} \oplus I_{V_2} \to I_{V_1} + I_{V_2} \to 0$$

we also can get the Hilbert function of $R/(I_{V_1} + I_{V_2})$.

(b) Using induction on the codimension, we construct our arithmetically Gorenstein schemes X which are not complete intersections in general. They have the following properties.
   (i) They are *generalized stick figures.* This means that they are the reduced union of linear varieties of codimension c (say), and no three components meet in a linear variety of codimension $c + 1$. In the case of curves, this is precisely the notion of a *stick figure.*
   There are several advantages to using generalized stick figures for X. First, there are many possible subconfigurations that we can link using X, if we can just devise a way to find the "right" ones. Second, any such link is guaranteed to be geometric, since X is reduced. Third, after making such a link and finding the sum of the linked ideals, the result is guaranteed to be reduced, thanks to the fact that it is a generalized stick figure! (This idea was used earlier in [44] for the case of CI-linked stick figure curves in $\mathbb{P}^3$.)
   (ii) Their Hilbert functions are "maximal" with a flat part in the middle. They are constructed inductively as a sum of G-linked ideals, by finding a suitable subset with "big" Hilbert function, which in turn is constructed by Basic Double G-Linkage. For example, here are the h-vectors of the arithmetically Gorenstein schemes in low codimension:

$$\begin{array}{lcccccccccccccc}
 & & & & & & \multicolumn{4}{c}{\overbrace{\hspace{6cm}}^{\text{flat}}} & & & & & \\
\text{codim 2:} & 1 & 2 & 3 & \dots & t-1 & t & t & \dots & t & t-1 & \dots & 3 & 2 & 1 \\
\text{codim 3:} & 1 & 3 & 6 & \dots & \binom{t}{2} & \binom{t+1}{2} & \binom{t+1}{2} & \dots & \binom{t+1}{2} & \binom{t}{2} & \dots & 6 & 3 & 1 \\
\text{codim 4:} & 1 & 4 & 10 & \dots & \binom{t+1}{3} & \binom{t+2}{3} & \binom{t+2}{3} & \dots & \binom{t+2}{3} & \binom{t+1}{3} & \dots & 10 & 4 & 1
\end{array}$$

(c) The schemes X obtained in (b) will be used to link. We will assume that $\operatorname{codim} X = c - 1$ and construct our schemes in codimension c. Suppose that a desired SI-sequence $\underline{h}$ is given. We use the formula of part (a) to work backwards, to determine the Hilbert function of an arithmetically Cohen-Macaulay subconfiguration $V_1 \subset X$ that would be needed to produce $\underline{h}$ as a sum of linked ideals.

(d) We use our knowledge of the schemes X to prove that an arithmetically Cohen-Macaulay scheme $V_1 \subset X$, as described in (c), in fact does exist. This is the most technical part of the proof.

11.2. **Smooth curves in $\mathbb{P}^3$.**

A. A long-standing problem, with many subtle variations, was to determine the possible pairs (d, g) of degree and genus of smooth curves in $\mathbb{P}^3$ (or $\mathbb{P}^n$). This was solved by Gruson and Peskine [45] for curves in $\mathbb{P}^3$ and by Rathmann [97] for curves in $\mathbb{P}^4$ and $\mathbb{P}^5$. Substantial progress has been made by Chiantini, Ciliberto and Di Gennaro [28] in higher projective spaces.

One variation of this problem is to determine a bound for the (arithmetic) genus of a non-degenerate, integral, degree d curve $C \subset \mathbb{P}^3$ lying on an irreducible surface S of degree k, and to describe the extremal curves. This problem was solved by Harris [47], who gave a specific bound. Furthermore, he showed that the curves which are extremal with respect to this bound are precisely the curves residual to a plane curve via certain complete intersections. Note that they are thus arithmetically Cohen-Macaulay. (A deeper problem is to bound the genus of a smooth curve in $\mathbb{P}^3$ not lying on any surface of degree $< k$. There is much progress on this problem, beginning with work of Hartshorne and Hirschowitz [54].)

B. Harris' work mentioned above used the Hilbert function of the general hyperplane section of the curve C. He showed that the general hyperplane section must have the Uniform Position Property (see Definition 4.8). (Note that Harris' proof of the uniform position property for a general hyperplane section required characteristic zero. It has been proved in characteristic p for $\mathbb{P}^n$, $n \geq 4$, by Rathmann [97].) This led to natural questions:

Q1. What are all the possible Hilbert functions for the general hyperplane section of an integral curve in $\mathbb{P}^3$? (Same question for $\mathbb{P}^n$.)

Q2. What are all the possible Hilbert functions for the general hyperplane section of an integral arithmetically Cohen-Macaulay curve in $\mathbb{P}^3$? (Same question for $\mathbb{P}^n$.)

Q3. What are all the possible Hilbert function of sets of points in $\mathbb{P}^2$ with the Uniform Position Property? (Same question for $\mathbb{P}^{n-1}$.)

Q4. Do the questions above (for fixed n) have the same answer?

The answer to these questions is known for $n = 3$, but open otherwise (see also Section 11.4). The answer to Q4 is "yes" when $n = 3$, and the Hilbert functions that arise are those of so-called *decreasing type*. This means the following. Let Z be the set of points (either the hyperplane section of an integral curve or a set of points with the Uniform Position Property). Then the Hilbert function of the Artinian reduction, A, of R/I_Z looks as follows. Let d_1 be the degree of the first minimal generator of I_Z, and d_2 the degree of the second. Note that $d_1 \leq d_2$. Let r be the Castelnuovo-Mumford regularity of $\mathcal{I}_Z$. Then

$$h_A(t) = \begin{cases} t+1 & \text{if } t < d_1 \\ d_1 & \text{if } d_1 \leq t \leq d_2 - 1 \\ \text{(strictly decreasing)} & \text{if } d_2 - 1 \leq t \leq r \\ 0 & \text{if } t \geq r \end{cases}$$

Work on this problem was carried out in [45], [67], [98]. The interesting part is to construct an integral arithmetically Cohen-Macaulay curve with the desired h-vector, and this was done in [67] by a nice application of liaison. A completely different approach, using lifting techniques, was carried out in [29].

11.3. **Smooth surfaces in $\mathbb{P}^4$, smooth threefolds in $\mathbb{P}^5$.** In the classification of smooth codimension two subvarieties (and by Hartshorne's conjecture, we stop with threefolds in $\mathbb{P}^5$), it has typically been the case that adjunction theory or other methods have been used to narrow down the possibilities (see for instance [8]), and then liaison has been used to construct examples.

We give an illustration of this idea by sketching a result of Miró-Roig from [81]. A natural question is to determine the degrees d for which there exists a smooth, non-arithmetically Cohen-Macaulay threefold in $\mathbb{P}^5$. It had been shown by Bănică [5] that such threefolds exist for any odd $d \geq 7$ and for any even $d = 2k > 8$ with $k = 5s+1$, $5s+2$, $5s+3$ or $5s+4$. It had been shown by Beltrametti, Schneider and Sommese [7] that any smooth threefold in $\mathbb{P}^5$ of degree 10 is arithmetically Cohen-Macaulay.

It remained to consider the case where $d = 10n$, $n \geq 2$. Miró-Roig proved the existence of such threefolds using liaison. Her idea was to begin with well-known non-arithmetically Cohen-Macaulay threefolds in $\mathbb{P}^5$ and use the fact that the property of being arithmetically Cohen-Macaulay is preserved under liaison. In addition, she used the following result of Peskine and Szpiro [91] to guarantee smoothness:

Theorem 11.2. *Let $X \subset \mathbb{P}^n$, $n \leq 5$, be a local complete intersection of codimension two. Let m be a twist such that $\mathcal{I}_X(m)$ is globally generated. Then for every pair $d_1, d_2 \geq m$ there exist forms $F_i \in H^0(\mathcal{I}_X(d_i))$, $i = 1, 2$, such that the corresponding hypersurfaces V_1 and V_2 intersect properly and link X to a variety X'. Furthermore, X' is a local complete intersection with no component in common with X, and X' is nonsingular outside a set of positive codimension in Sing X.*

(This special case of the theorem is quoted from [33], Theorem 2.1.) Miró-Roig considered an arithmetically Buchsbaum threefold Y with locally free resolution

$$0 \to \mathcal{O}_{\mathbb{P}^5} \oplus \mathcal{O}_{\mathbb{P}^5}(1)^3 \to \Omega^1(3) \to \mathcal{I}_Y(6) \to 0$$

(see also Example 7.14). Since $\mathcal{I}_Y(6)$ is globally generated, Theorem 11.2 applies. Linking by two general hypersurfaces of degrees 6 and 7, respectively, she obtains a smooth residual threefold X of degree 30, and using the mapping cone construction she obtains the locally free resolution of $\mathcal{I}_X$. Playing the same kind of game, she is able to obtain from X smooth threefolds of degrees $10n$, $n \geq 5$, by linking X using hypersurfaces of degree 10 and $n+3$. The remaining cases, degrees 20 and 40, are obtained by similar methods, starting with different Y.

11.4. **Hilbert function questions.** We have seen above that liaison is useful for showing the existence of interesting objects. In this section we will see that liaison can sometimes be used to prove non-existence results, as well as results which reduce the possibilities. For instance, we consider the question of describing the possible Hilbert functions of sets of points in $\mathbb{P}^3$ with the Uniform Position Property.

Example 11.3. Does there exist a set of points in $\mathbb{P}^3$ with the Uniform Position Property and h-vector

$$1 \quad 3 \quad 6 \quad 5 \quad 6,$$

and if so, what can we say about it? Suppose that such a set, Z, does exist. Note that the growth in the h-vector from degree 3 to degree 4 is maximal, according to Macaulay's growth condition [66]. This implies, thanks to [11] Proposition 2.7, that the components $[I_Z]_3$ and $[I_Z]_4$ both have a GCD of degree 1, defining a plane H. It also follows using the same argument as [11] Example 2.11 that Z consists of either 14 or 15 points on H, plus 6 or 7 points not on H (of which 4 or 5 are on a line). Such a Z clearly does not have the Uniform Position Property!

Example 11.4. Does there exist a set of points in $\mathbb{P}^3$ with the Uniform Position Property and h-vector

$$1 \quad 3 \quad 6 \quad 5 \quad 5,$$

and if so, what can we say about it? Let Z be such a set. In this case we do not have maximal growth from degree 3 to degree 4, but we again consider the component in degree 3. This time we will not have a GCD, but we can consider the base locus of the linear system $|[I_Z]_3|$. Suppose that this base locus is zero-dimensional. Then three general elements of $[I_Z]_3$ give a complete intersection, $I_X = (F_1, F_2, F_3)$. This means that Z is linked by X to a zeroscheme W, and we can make a Hilbert function (h-vector) calculation (cf. Corollary 5.11 and Example 5.15 (c)):

degree	0	1	2	3	4	5	6	7
R/I_X	1	3	6	7	6	3	1	0
R/I_Z	1	3	6	5	5	0	0	0
R/I_W	0	0	0	2	1	3	1	0

This means that the residual, W, has h-vector 1 3 1 2, which is impossible (it violates Macaulay's growth condition).

Thus we are naturally led to look for an example consisting of a set of 20 general points, Z, on an irreducible curve C of degree 5. (We do not justify this, although similar considerations can be found in the proof of Theorem 4.7 of [11], but we hope that it is clear that this is the natural place to look, even if it is not clear that it is the *only* place to look.) The Hilbert function of Z has to agree with that of Z up to degree 4. One can check that a general curve C of degree 5 and genus 1 will do the trick (and no other will). Hence the desired set of points does exist.

11.5. **Arithmetically Buchsbaum curves specialize to stick figures.** We have seen how to use Liaison Addition to construct minimal arithmetically Buchsbaum curves (Example 6.10) and how to use the Lazarsfeld-Rao property to give all the possible (d, g) combinations possible for arithmetically Buchsbaum curves (Remark 9.7). Now we sketch how these ideas were refined in [19] and applied to show that every arithmetically Buchsbaum specializes to a stick figure. This is a special case of the Zeuthen problem, a long-standing problem that was solved a few years ago by Hartshorne [51]. The general question is whether every smooth curve in $\mathbb{P}^3$ specializes to a stick figure, and Hartshorne showed that the answer is "no." This makes it more interesting that the answer is "yes" for arithmetically Buchsbaum curves.

Let C be an arithmetically Buchsbaum curve. The basic idea here is that the Lazarsfeld-Rao property provides the desired deformation, if we can produce a stick figure using basic double links which is cohomologically the same as C. So there are two parts to the story.

First we have to produce a minimal element which is a stick figure, and second we have to study basic double links and show that we can always keep producing stick figures.

For the first part, it is a refinement of the construction given in Example 6.10. Skipping details, we merely note here that if C_1 and C_2 are both pairs of skew lines chosen generically, then F_1 and F_2 can be chosen to be unions of planes, and for a sufficiently general choice, the curve C constructed by Liaison Addition will be a stick figure. To see that this procedure can give a minimal element for *any* Buchsbaum even liaison class is somewhat more technical, but is an extension of this idea.

For the second part, recall that a basic double link is obtained by starting with a curve C and a surface F containing C, and taking the union Y of C and a general hyperplane section of F. If C is a union of lines and F is a union of planes then clearly Y will also be a union of lines. The first problem is to show that we can always arrange that there exists a surface F which is a union of planes. For instance, if C is a union of ≥ 3 skew lines on a quadric surface (this is not arithmetically Buchsbaum, but gives the idea), and if we want $\deg F = 2$, then F clearly cannot be chosen to be a union of planes. So we have to show that a union of planes can always be obtained in our case. But there is a more subtle problem.

For example, suppose that C is a set of two skew lines, and suppose that we make a sequence of three basic double links using F_1, F_2 and F_3 of degrees 20, 15 and 4 respectively, obtaining curves Y_1, Y_2 and Y_3 of degrees 22, 37 and 41 respectively. A little thought shows that one cannot avoid that Y_3 have a triple point! (The key is that $\deg F_1 > \deg F_2 > \deg F_3$.) Thus this sequence of basic double links *cannot* yield a stick figure.

The solution to this dilemma is to show that there is a *cohomologically equivalent* sequence of basic double links using surfaces G_1, G_2, G_3 with $\deg G_1 \leq \deg G_2 \leq \deg G_3$. Then the type of problem described in the last paragraph does not occur. Again, the details are technical, and we refer the reader to [18] and [19].

11.6. **The minimal free resolution of generic forms.** An important problem, variations of which have been studied by many people, is to describe the Hilbert function or minimal free resolution of an ideal $I \subset R = K[x_1, \ldots, x_n]$ generated by a general set of forms of fixed degrees (not necessarily all the same). The answer to the Hilbert function problem has been conjectured by Fröberg and we will not describe it here. It is known to hold when $n \leq 3$ and when the number of generators is $n + 1$.

For the minimal free resolution, the answer has been conjectured by Iarrobino. At the heart of this is the idea that if the forms are general then there should be no "ghost terms" in the minimal free resolution, i.e. there should be no summand $R(-t)$ that appears in consecutive free modules in the resolution. One can see immediately that this is too optimistic, however. For instance, if I has two generators of degree 2 and one of degree 4 then there is a term $R(-4)$ corresponding to a first syzygy and a term $R(-4)$ corresponding to a generator. So the natural conjecture is that apart from such terms which are forced by Koszul relations, there should be no ghost terms.

This was proved to be false in [74]. A simple counterexample is the case of four generators in $K[x_1, x_2, x_3]$ of degrees 4,4,4 and 8. The minimal free resolution turns out to be

$$0 \to \begin{pmatrix} R(-10) \\ \oplus \\ R(-11)^2 \end{pmatrix} \to \begin{pmatrix} R(-8)^3 \\ \oplus \\ R(-9)^2 \\ \oplus \\ R(-10) \end{pmatrix} \to \begin{pmatrix} R(-4)^3 \\ \oplus \\ R(-8) \end{pmatrix} \to R \to R/I \to 0$$

The term $R(-8)$ that does not split arises from Koszul relations, as above, but the summand $R(-10)$ shared by the second and third modules also does not split and this does *not* arise from Koszul relations.

The paper [74] made a general study of the minimal free resolution of $n+1$ general forms in R. The minimal free resolution was obtained in many cases (depending on the degrees of the generators) and the main tools were liaison and a technical lemma from [77] giving a bound on the graded Betti numbers for Gorenstein rings. The key to this work is Corollary 5.19 above, which says that our ideal I can always be directly linked to a Gorenstein ideal.

Here is the basic idea. Knowing the Hilbert function for the $n+1$ general forms leads to the Hilbert function of the linked Gorenstein ideal. The technical lemma of [77] then gives good bounds for the graded Betti numbers of the linked Gorenstein ideal, and in fact these bounds can often be shown to be sharp. Then the mapping cone obtained from the first sequence of Lemma 5.10 can be used to give a free resolution of R/I. One can then determine to what extent this resolution is minimal. In particular, ghost terms in the minimal free resolution of the Gorenstein ideal translate to ghost terms in the minimal free resolution of I. Especially when $n = 3$, we can often arrange ghost terms for the Gorenstein ideal (thanks to the Buchsbaum-Eisenbud structure theorem [22] and the work of Diesel [34]).

12. Open Problems

In this section we collect the open questions that were mentioned in the preceding sections, and add some more.

1. Describe the Hilbert functions for general hyperplane sections of integral curves in $\mathbb{P}^n$ ($n \geq 4$) and for sets of points in $\mathbb{P}^{n-1}$ with the Uniform Position Property. (See the discussion starting on page 59.)
2. Find a description of all the possible Hilbert functions of Artinian Gorenstein graded K-algebras. Find a description of all the possible Hilbert functions of reduced arithmetically Gorenstein subschemes of $\mathbb{P}^n$. (Is the answer to this last question precisely the SI-sequences?)
3. Classify the possible graded Betti numbers for Gorenstein algebras in codimension ≥ 4. See Questions 4.14 and 4.15 and the discussion following them.
4. It is an old problem (see e.g. [49] Exer. 2.17 (d)) whether every irreducible curve $C \subset \mathbb{P}^3$ is a set-theoretic complete intersection. It is not true that a curve which is a set-theoretic complete intersection must be arithmetically Cohen-Macaulay (see e.g. [96]). However, the first author has conjectured that such a curve must be linearly normal. Some progress in this direction was achieved by Jaffe [59]. In the first draft

of these notes we made the comment here that we were not aware even of a curve which is a set-theoretic complete intersection but is not self-linked. However, R. Hartshorne has provided us with an example, which we have recorded in Example 12.1 below.

5. Find conditions that are necessary and sufficient for two schemes in codimension ≥ 3 to be evenly CI-linked or evenly G-linked.
6. In particular, is it true that two arithmetically Cohen-Macaulay schemes of the same codimension are G-linked in finitely many steps? As an important first case, is it true that two arithmetically Gorenstein subschemes of the same codimension are G-linked in finitely many steps?
7. Extend the known CI-liaison invariants for arithmetically Cohen-Macaulay subschemes (cf., e.g., Theorem 10.1) to non-arithmetically Cohen-Macaulay subschemes which allow one to distinguish CI-liaison classes within an even G-liaison class of a non-arithmetically Cohen-Macaulay subscheme.
8. Compare the equivalence relations generated by geometric G-liaison and (algebraic) G-liaison on the set of subschemes of $\mathbb{P}^n$ having codimension c and being generically Gorenstein. (cf. Section 10 for results in the case of CI-liaison.)
9. Find a structure theorem similar to the LR-property that holds for G-liaison or for CI-liaison in higher codimension.
10. Establish upper and lower bounds for the various minimal shifts attached to an equidimensional scheme (cf. Remark 9.4).
11. Find a good concept for minimal elements in an even G-liaison class (cf. Section 9).
12. Find conditions like the theorem of Peskine and Szpiro [91] (cf. Theorem 11.2) which guarantee that a G-linked residual scheme is smooth (in the right codimension). Find applications of this to the classification of smooth codimension 3 subschemes. See [80] for more on this idea.

Example 12.1. In an earlier draft of these notes we asked if there is any smooth curve in $\mathbb{P}^3$ which is a set-theoretic complete intersection but not self-linked. We believed that there should be such a curve, but were not aware of one. This example is due to Robin Hartshorne, who has kindly allowed us to reproduce it here.

A curve is self-linked if it is a set-theoretic complete intersection of multiplicity 2. So here we will construct, for every integer $d > 0$, a smooth curve in $\mathbb{P}^3$ that is set-theoretically the complete intersection of multiplicity d, but of no lower multiplicity.

Start with a smooth plane curve of degree d, having a d-fold inflectional tangent at a point P. Let X be the cone over that curve in $\mathbb{P}^3$. Let L be the cone over P. Then L is a line on X, dL is a complete intersection on X, and no lower multiple of L is a complete intersection of X with another surface. Now let C be a smooth curve in the linear system $|L + mH|$ on X, for $m \gg 0$.

Note that dC is linearly equivalent to $dL + mdH = (md + 1)H$. Therefore dC is the intersection of X with another surface in $\mathbb{P}^3$, and so C is a set-theoretic complete intersection of multiplicity d. Note that no smaller multilple of C is the complete intersection of X with anything else, because eC for $e < d$ is not a Cartier divisor on X. But could eC be an intersection of two other surfaces? Since C has degree $md + 1$, if F is any other surface containing C, then the degree of $X \cdot F$ is $d \cdot \deg F$, so $\deg F > m$. So if C

is the set-theoretic complete intersection of F and G, then $\deg F \cdot G$ is $> m^2$, and the multiplicity of the structure on C is $> m^2/(md+1)$, which for $m \gg 0$ is $> d$. (In fact, to obtain $m^2/(md+1) > d$, i.e. $m(m-d^2) > d$, it is enough to take $m > d^2$.)

References

[1] M. Amasaki, *On the Structure of Arithmetically Buchsbaum Curves in* $\mathbb{P}^3$, Publ. Res. Inst. Math. Sci. **20** (1984), 793–837.

[2] M. Amasaki, *Existence of homogeneous ideals fitting into long Bourbaki sequences*, Proc. Amer. Math. Soc. **27** (1999), 3461–3466.

[3] M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc. **94** (1969).

[4] E. Ballico, G. Bolondi and J. Migliore, *The Lazarsfeld-Rao Problem for Liaison Classes of Two-Codimensional Subschemes of* $\mathbb{P}^n$, Amer. J. Math. **113** (1991), 117–128.

[5] C. Bănică, *Smooth reflexive sheaves*, Proceedings of the Colloquium on Complex Analysis and the Sixth Romanian-Finnish Seminar, Rev. Roumaine Math. Pures Appl. 36 (1991), no. 9-10, 571–593.

[6] D. Bayer, "The division algorithm and the Hilbert scheme," Ph.D. thesis, Harvard University, Department of Mathematics, June 1982, order number 82-22588, University Microfilms International, 300 N. Zeeb Rd., Ann Arbor, MI 48106.

[7] M. C. Beltrametti, M. Schneider and A. Sommese, *Threefolds of degree 9 and 10 in* $\mathbb{P}^5$, Math. Ann. **288** (1990), 413–444.

[8] M. Beltrametti and A. Sommese, "The Adjunction Theory of Complex Projective Varieties," De Gruyter, Expositions in Mathematics 16 (1995).

[9] V. Beorchia and Ph. Ellia, *Normal bundle and complete intersections*, Rend. Sem. Mat. Univ. Politec. Torino **48** (1990), 553–562.

[10] D. Bernstein and A. Iarrobino, *A nonunimodal graded Gorenstein Artin algebra in codimension five*, Comm. Alg. **20(8)** (1992), 2323–2336.

[11] A. Bigatti, A. V. Geramita and J. Migliore, *Geometric Consequences of Extremal Behavior in a Theorem of Macaulay*, Trans. Amer. Math. Soc. **346** (1994), 203–235.

[12] C. Bocci and G. Dalzotto, *Gorenstein Points in* $\mathbb{P}^3$, preprint.

[13] C. Bocci, G. Dalzotto, R. Notari and M. L. Spreafico, in preparation.

[14] M. Boij, *Graded Gorenstein Artin algebras whose Hilbert functions have a large number of valleys*, Comm. Algebra **23(1)** (1995), 97–103.

[15] M. Boij and D. Laksov, *Nonunimodality of Graded Gorenstein Artin Algebras*, Proc. Amer. Math. Soc. **120** (1994), 1083–1092.

[16] G. Bolondi, *Irreducible families of curves with fixed cohomologies*, Arch. Math. **53** (1989), 300–305.

[17] G. Bolondi and J. Migliore, *Buchsbaum Liaison Classes*, J. Algebra **123** (1989), 426–456.

[18] G. Bolondi and J. Migliore, *The Structure of an Even Liaison Class*, Trans. Amer. Math. Soc. **316** (1989), 1–37.

[19] G. Bolondi and J. Migliore, *Configurations of Linear Projective Subvarieties*, in "Algebraic Curves and Projective Geometry, Proceedings (Trento, 1988)," Lecture Notes in Mathematics, vol. 1389, Springer–Verlag (1989), 19–31.

[20] G. Bolondi and J. Migliore, *The Lazarsfeld-Rao property on an arithmetically Gorenstein variety*, manuscripta math. 78 (1993), 347–368.

[21] M. Brodmann and U. Nagel, *Bounding cohomological Hilbert functions by hyperplane sections*, J. Algebra **174** (1995), 323–348.

[22] D. Buchsbaum and D. Eisenbud, *Algebra Structures for Finite Free Resolutions, and some Structure Theorems for Ideals of Codimension 3*, Amer. J. of Math. **99** (1977), 447–485.

[23] R. O. Buchweitz and B. Ulrich, *Homological properties which are invariant under linkage*, Preprint, 1983.

[24] M. Casanellas, "Teoria de liaison en codimensió arbitrària," Ph.D. Thesis, University of Barcelona, 2001.

[25] M. Casanellas and R. M. Miró-Roig, *Gorenstein Liaison of curves in* $\mathbb{P}^4$, J. Algebra **230** (2000), 656–664.

[26] M. Casanellas and R. M. Miró-Roig, *Gorenstein Liaison of divisors on standard determinantal schemes and on rational normal scrolls*, J. Pure Appl. Algebra **164** (2001), 325–343.

[27] G. Casnati and F. Catanese, *Quadratic sheaves and self-linkage*, preprint 2001.

[28] L. Chiantini, C. Ciliberto and V. Di Gennaro, *The genus of projective curves*, Duke Math. J. **70** (1993), no. 2, 229–245.

[29] L. Chiantini and F. Orecchia, *Plane Sections of Arithmetically Normal Curves in* $\mathbb{P}^3$, in "Algebraic Curves and Projective Geometry, Proceedings (Trento, 1988)," Lecture Notes in Mathematics, vol. 1389, Springer–Verlag (1989), 32–42.

[30] D. Cox, J. Little and D. O'Shea, "Ideals, Varieties and Algorithms," Undergraduate Texts in Mathematics, Springer-Verlag, 1992.

[31] E. Davis, A. V. Geramita and F. Orecchia, *Gorenstein Algebras and the Cayley-Bacharach Theorem*, Proc. Amer. Math. Soc. **93** (1985), 593–597.

[32] W. Decker, L. Ein and F. Schreyer, *Constructions of surfaces in* $\mathbb{P}^4$, J. Algebraic Geom. **2** (1993), 185–237.

[33] W. Decker and S. Popescu, *On Surfaces in* $\mathbb{P}^4$ *and 3-folds in* $\mathbb{P}^5$, in: Vector bundles in algebraic geometry (Durham, 1993), 69–100, London Math. Soc. Lecture Note Ser., 208, Cambridge Univ. Press, Cambridge, 1995.

[34] S. Diesel, *Irreducibility and Dimension Theorems for Families of Height 3 Gorenstein Algebras*, Pacific J. Math. **172** (1996), no. 2, 365–397.

[35] D. Eisenbud, "Commutative Algebra with a View toward Algebraic Geometry," Graduate Texts in Mathematics **150**, Springer-Verlag, 1995.

[36] E. G. Evans and P. A. Griffith, *Local cohomology modules for normal domains*, J. London Math. Soc. **22** (1979), 277–284.

[37] E. G. Evans and P. A. Griffith, "Syzygies", Cambridge University Press, 1985.

[38] D. Franco, S. Kleiman and A. Lascu, *Gherardelli linkage and complete intersections*, Michigan Math. J. **48** (2000), 271–279.

[39] A. Galligo, *A propos du théorème de préparation de Weierstrass*, in "Fonctions des Plusieurs Variables Complexes," Lecture Notes in Mathematics, vol. 409, Springer–Verlag (1974), 543–579.

[40] A. V. Geramita, T. Harima and Y.S. Shin, *Extremal Point Sets and Gorenstein Ideals*, Adv. Math. **152** (2000), 78–119.

[41] A. V. Geramita and J. Migliore, *On the Ideal of an Arithmetically Buchsbaum Curve*, J. Pure Appl. Algebra **54** (1988), 215–247.

[42] A. V. Geramita and J. Migliore, *Generators for the Ideal of an Arithmetically Buchsbaum Curve*, J. Pure Appl. Algebra **58** (1989), 147–167.

[43] A. V. Geramita and J. Migliore, *A Generalized Liaison Addition*, J. Algebra 163 (1994), 139–164.

[44] A. V. Geramita and J. Migliore, *Reduced Gorenstein Codimension Three Subschemes of Projective Space*, Proc. Amer. Math. Soc. **125** (1997), 943-950.

[45] L. Gruson and C. Peskine, *Genre des Courbes de L'Espace Projectif*, in "Proceedings, Algebraic Geometry (Tromsø, 1977)," Lecture Notes in Mathematics, vol. 687, Springer–Verlag (1978), 31–59.

[46] T. Harima, *Characterization of Hilbert functions of Gorenstein Artin algebras with the Weak Stanley property*, Proc. Amer. Math. Soc. **123** (1995), 3631–3638.

[47] J. Harris, *The Genus of Space Curves*, Math. Ann. **249** (1980), 191–204.

[48] J. Harris and D. Eisenbud, "Curves in Projective Space," Sem. de Mathématiques Supérieures–Université de Montréal (1982).

[49] R. Hartshorne, "Algebraic Geometry," Springer-Verlag, Graduate Texts in Mathematics **52**, 1977.

[50] R. Hartshorne, *Generalized Divisors on Gorenstein Schemes*, K-Theory **8** (1994), 287–339.

[51] R. Hartshorne, *Families of curves in* $\mathbf{P}^3$ *and Zeuthen's problem*, Mem. Amer. Math. Soc. **130** (1997), No. 617.

[52] R. Hartshorne, *Experiments with Gorenstein liaison*, Le Matematiche, Special volume in honor of Silvio Greco (to appear).

[53] R. Hartshorne, *Some examples of Gorenstein liaison in codimension three*, Collectanea Mathematica **53** (2002), 21–48.

[54] R. Hartshorne and A. Hirschowitz, *Nouvelles courbes de bon genre dans l'espace projectif.* Math. Ann. **280** (1988), no. 3, 353–367.

[55] G. Horrocks, *Vector bundles on the punctured spectrum of a local ring*, Proc. London Math. Soc. **14** (1964), 689–713.

[56] C. Huneke, *Numerical invariants of liaison classes*, Invent. Math. **75** (1984), 301–325.

[57] C. Huneke and B. Ulrich, *The structure of linkage*, Ann. Math. **126** (1987), 221–275.

[58] C. Huneke and B. Ulrich, *General Hyperplane Sections of Algebraic Varieties*, J. Algebraic Geom. **2** (1993), 487–505.

[59] D. Jaffe, *Applications of iterated curve blow-up to set theoretic complete intersections in* $\mathbb{P}^3$, J. Reine Angew. Math. **464** (1995), 1–45.

[60] S. Kleiman and B. Ulrich, *Gorenstein algebras, symmetric matrices, self-linked ideals, and symbolic powers*, Trans. Amer. Math. Soc. **349** (1997), 4973–5000.

[61] J. Kleppe, J. Migliore, R. M. Miró-Roig, U. Nagel and C. Peterson, *Gorenstein Liaison, Complete Intersection Liaison Invariants and Unobstructedness*, Mem. Amer. Math. Soc. **154** (2001), No. 732.

[62] M. Kreuzer, *On 0-dimensional complete intersections*, Math. Ann. **292** (1992), 43-58.

[63] R. Lazarsfeld and P. Rao, *Linkage of General Curves of Large Degree*, in "Algebraic Geometry–Open Problems (Ravello, 1982)," Lecture Notes in Mathematics, vol. 997, Springer–Verlag (1983), 267–289.

[64] J. Lesperance, *Gorenstein liaison of some curves in* $\mathbb{P}^4$, Collectanea Mathematica **52** (2001), 219–230.

[65] J. Lesperance, *Gorenstein liaison of degenerate Buchsbaum curves in* $\mathbb{P}^4$, in preparation.

[66] F.S. Macaulay, *Some properties of enumeration in the theory of modular systems*, Proc. Lond. Math. Soc. **26** (1927), 531–555.

[67] R. Maggioni and A. Ragusa, *The Hilbert Function of Generic Plane Sections of Curves in* $\mathbb{P}^3$, Inv. Math. **91** (1988), 253–258.

[68] M. Martin-Deschamps and D. Perrin, *Sur la Classification des Courbes Gauches*, Astérisque 184–185, Soc. Math. de France (1990).

[69] J. Migliore, *On Linking Double Lines*, Trans. Amer. Math. Soc. **294** (1986), 177–185.

[70] J. Migliore, *Geometric invariants for liaison of space curves*, J. Algebra **99** (1986), 548–572.

[71] J. Migliore, *Liaison of a Union of Skew Lines in* $\mathbb{P}^4$, Pacific J. of Math. **130** (1987), 153–170.

[72] J. Migliore, "Introduction to Liaison Theory and Deficiency Modules," Progress in Mathematics **165**, Birkhäuser, 1998.

[73] J. Migliore, *Gorenstein liaison via divisors*, In: Proceedings of the Workshop on "Free Resolution of Defining Ideals of Projective Varieties" held at the Research Institute for Mathematical Sciences, Kyoto University, Kyoto, July 1998. Surikaisekikenkyusho Kokyuroku (1999).

[74] J. Migliore and R. Miró-Roig, *On the Minimal Free Resolution of* $n+1$ *generic forms*, preprint.

[75] J. Migliore and U. Nagel, *Lifting Monomial Ideals*, Comm. Algebra **28** (2000), Special volume in honour of R. Hartshorne, 5679-5701.

[76] J. Migliore and U. Nagel, *Monomial ideals and the Gorenstein liaison class of a complete intersection*, Compositio Math. (to appear).

[77] J. Migliore and U. Nagel, *Reduced arithmetically Gorenstein schemes and Simplicial Polytopes with maximal Betti numbers*, preprint.

[78] J. Migliore, U. Nagel and C. Peterson, *Buchsbaum-Rim sheaves and their multiple sections*, J. Algebra **219** (1999), 378-420.

[79] J. Migliore, U. Nagel and C. Peterson, *Constructing schemes with prescribed cohomology in arbitrary codimension*, J. Pure Appl. Algebra **152** (2000), 245–251.

[80] J. Migliore and C. Peterson, *A construction of codimension three arithmetically Gorenstein subschemes of projective space*, Trans. Amer. Math. Soc. **349** (1997), 3803-3821.

[81] R. Miró-Roig, *On the Degree of Smooth non-Arithmetically Cohen-Macaulay Threefolds in* $\mathbb{P}^5$, Proc. Amer. Math. Soc. **110** (1990), 311–313.
[82] R. Miró-Roig, *Gorenstein liaison*, Journal of Mathematics, University of Milano.
[83] R. Miró-Roig, *Complete intersection liaison and Gorenstein liaison: new results and open problems*, Le Matematiche, Special volume in honor of Silvio Greco (to appear).
[84] D. Mumford, "Lectures on curves on an algebraic surface," Ann. of Math. Studies **59**, Princeton Univ. Press, 1966.
[85] U. Nagel, *Even Liaison Classes Generated by Gorenstein Linkage*, J. Algebra **209** (1998), no. 2, 543–584.
[86] U. Nagel, *Eilenberg-MacLane modules*, in preparation.
[87] U. Nagel, *Characterization of projective subschemes by locally free resolutions*, in preparation.
[88] U. Nagel, R. Notari and M. L. Spreafico, *Even liaison classes of double lines and certain ropes*, preprint, 2001.
[89] U. Nagel, R. Notari and M. L. Spreafico, *On the even Gorenstein liaison class of ropes on a line*, Le Matematiche, Special volume in honor of Silvio Greco (to appear).
[90] S. Nollet, *Even Linkage Classes*, Trans. Amer. Math. Soc. **348** (1996), no. 3, 1137–1162.
[91] C. Peskine and L. Szpiro, *Liaison des variétés algébriques. I*, Invent. Math. **26** (1974), 271–302.
[92] A. Ragusa and G. Zappalà, *Properties of 3-codimensional Gorenstein schemes*, Comm. Algebra **29** (2001), 303–318.
[93] A. Ragusa and G. Zappalà, *Gorenstein schemes on general hypersurfaces of* $\mathbb{P}^r$, Nagoya Math. J. **162** (2001), 111–125.
[94] P. Rao, *Liaison among Curves in* $\mathbb{P}^3$, Invent. Math. **50** (1979), 205–217.
[95] P. Rao, *Liaison Equivalence Classes*, Math. Ann. **258** (1981), 169–173.
[96] P. Rao, *On Self-Linked Curves*, Duke Math. J. **49**(2) (1982), 251–273.
[97] J. Rathmann, *The genus of curves in* P^4 *and* P^5, Math. Z. **202** (1989), no. 4, 525–543.
[98] T. Sauer, *Smoothing Projectively Cohen-Macaulay Space Curves*, Math. Ann. **272** (1985), 83–90.
[99] P. Schenzel, *Notes on Liaison and Duality*, J. Math. of Kyoto Univ. **22** (1982), 485–498.
[100] P. Schenzel, "Dualisierende Komplexe in der lokalen Algebra und Buchsbaum-Ringe," LNM **907**, Springer-Verlag, 1982.
[101] P. Schwartau, *Liaison Addition and Monomial Ideals*, Ph.D. thesis, Brandeis University (1982).
[102] J.-P. Serre, *Faisceaux algebriques coherents*, Ann. Math. **61** (1955), 197–278.
[103] R. Strano, *A characterization of complete intersection curves in* $\mathbb{P}^3$, Proc. Amer. Math. Soc. **104** (1988), 711–715.
[104] R. Strano, *Biliaison classes of curves in* $\mathbb{P}^3$, Preprint, 2002.
[105] J. Stückrad and W. Vogel, "Buchsbaum rings and applications," Springer-Verlag, 1986.
[106] C. Walter, *Cohen-Macaulay liaison*, J. Reine Angew. Math. **444** (1993), 101–114.
[107] J. Watanabe, *A note on Gorenstein rings of embedding codimension 3*, Nagoya Math. J. **50** (1973), 227–232.
[108] J. Watanabe, *The Dilworth number of Artin Gorenstein rings*, Adv. Math. **76** (1989), 194–199.
[109] C. Weibel, "An introduction to homological algebra," Cambridge studies in advanced mathematics **38**, Cambridge University Press, 1994.

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