MA 361 - 05/05/2003 FINAL EXAM	Spring 2003 A. Corso	Name: ____________________ ____________________

PLEASE, BE NEAT AND SHOW ALL YOUR WORK; JUSTIFY YOUR ANSWER.

Problem Number	Possible Points	Points Earned
1	20	
2	20	
3	15	
4	15	
5	15	
6	15	
TOTAL	100	

1. (i) Compute the indicated product of cycles that are permutations of S_8
   - $(1,2)(4,7,8)(2,1)(7,2,8,1,5)$
   - $(1,4,5)(7,8)(2,5,7)$.

   (ii) Express the following permutation of S_8

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 4 & 7 & 2 & 5 & 8 & 6 \end{pmatrix}$$

   as product of disjoint cycles and then as product of transpositions.

   (iii) Consider the following permutations of S_6

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

   and compute

$$\tau\sigma, \qquad \tau^2\sigma, \qquad \sigma^{-1}\tau\sigma, \qquad \sigma^{100}, \qquad |\langle\tau^2\rangle|.$$

   (iv) Find the index of $\langle\sigma\rangle = (1,2,5,4)(2,3)$ in S_5.

pts: /20

2. (i) Draw the subgroup diagram for the subgroups of the group $\mathbb{Z}_{12}$.

   (ii) Find all cosets of the subgroup $\langle\overline{4}\rangle$ of $\mathbb{Z}_{12}$.

   (iii) Find the number of automorphisms of $\mathbb{Z}_{12}$.

   (iv) Find the order of $\overline{26}+\langle\overline{12}\rangle$ in $\mathbb{Z}_{60}/\langle\overline{12}\rangle$.

pts: /20

3. Choose one of the following problems:

   (a) Determine whether the map

$$\varphi\colon\ (M_2(\mathbb{R}), \cdot) \longrightarrow (\mathbb{R}, \cdot),$$

   where $\varphi(A) = \det(A)$, is an isomorphism of binary structures.

   Explain.

   (b) Let F be the set of all functions f mapping $\mathbb{R}$ into $\mathbb{R}$ that have derivatives of all orders. Determine whether the map

$$\varphi\colon\ (F, +) \longrightarrow (F, +),$$

   where $\varphi(f)(x) = \dfrac{d}{dx}\displaystyle\int_0^x f(t)\,dt$, is an isomorphism of binary structures.

   Explain.

pts: /15

4. Choose one of the following problems:

   (a) Let S be the set of all real numbers except -1. Define $*$ on S by

   $$a * b = a + b + ab.$$

   (i) Show that $*$ gives a binary operation on S.

   (ii) Show that $(S, *)$ is a group.

   (b) Let $\varphi\colon\ G \longrightarrow G'$ be an homomorphism of groups.

   If H is a subgroup of G, then $\varphi[H]$ is a subgroup of G'.

pts: /15

5. Choose one of the following problems:

   (a) Let H be a subgroup of a group G. For $a, b \in G$, let $a \sim b$ if and only if $ab^{-1} \in H$. Show that $\sim$ is an equivalence relation on G.

   (b) Let G be a group and suppose $a \in G$ generates a cyclic subgroup of order 2 and is the unique such element. Show that $ax = xa$ for all $x \in G$.

pts: /15

6. Choose one of the following problems:

   (a) A **torsion group** is a group all of whose elements have finite order. A group is **torsion free** if the identity is the only element of finite order.

   Prove that the torsion subgroup T of an abelian group G is a normal subgroup of G, and that G/T is torsion free.

   (b) Show that the inner automorphisms of a group G form a normal subgroup of all automorphisms of G under compositions.

pts: $\quad$ /15