THE EQUIVARIANT DOLD-THOM THEOREM

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1. References

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2. The classical Dold-Thom Theorem

We begin by recalling the infinite symmetric product construction on (based) topological spaces. Let X be a space. Then the n-fold cartesian product X^n has a natural action of the symmetric group on n letters Σ_n. We define the **n-fold symmetric product** to be the quotient

$$\mathrm{Symm}^n(X) = (X^n)/\Sigma_n.$$

If X has a chosen basepoint $* \in X$, then for each n we have a natural inclusion $\mathrm{Symm}^n(X) \hookrightarrow \mathrm{Symm}^{n+1}(X)$ given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, *)$. We then define the **infinite symmetric product** on X to be

$$\mathrm{Symm}^\infty(X, *) = \operatorname*{colim}_n \mathrm{Symm}^n(X).$$

Note that $\mathrm{Symm}^\infty(X, *)$ is a commutative topological monoid, with unit element given by $*$. In fact, the symmetric product on X is the *free* commutative monoid on X.

The classical Dold-Thom theorem reads

Theorem 1 (Dold-Thom,'58)**.** *Let X be a pointed, connected CW-complex. Then there is a weak equivalence*

$$\mathrm{Symm}^\infty(X, *) \xrightarrow{\sim} \prod_{n\geq 1} K\big(\mathrm{H}_n(X;\mathbb{Z}), n\big).$$

In other words, $\pi_n(\mathrm{Symm}^\infty(X, *)) \cong \mathrm{H}_n(X;\mathbb{Z})$ for $n \geq 1$.

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Remark 1. One can remove the connectedness assumption on X by introducing a "group-completion" of the infinite symmetric product. One then gets the formula

$$\pi_n(\mathrm{Symm}^\infty(X,*)^+) \cong \tilde{\mathrm{H}}_n(X;\mathbb{Z})$$

for all $n \geq 0$.

Remark 2. This theorem should not be surprising if one recalls the definition of singular homology. Given a space X, the singular homology of X is given by the homotopy groups of the free simplicial abelian group on the singular complex $S_\bullet(X)$. Thus already in the definition of homology we are taking homotopy groups of the "free abelian group" construction. This fits in with Quillen's slogan "Homology is the left derived functor of abelianization."

Remark 3. We will also write $\mathbb{N}(X,*)$ or $\tilde{\mathbb{N}}(X)$ or $\mathbb{N}\otimes X$ for the infinite symmetric product. As this is a commutative topological monoid, the usual Grothendieck group construction produces a topological abelian group, which we will denote $\tilde{\mathbb{Z}}(X)$ or $\mathbb{Z}\otimes X$.

One can in fact produce, given any abelian group A, a topological abelian group $A\otimes X$ with the property that

$$\pi_n(A\otimes X) \cong \tilde{\mathrm{H}}_n(X;A).$$

A construction of this space, due to McCord, is as follows. Let $B_n(A,X)$ be the set of functions $f : X \to A$ such that $f(*) = 0$ and which are non-zero at at most n points. For any $a \in A$ and $x \in X$, let a_x be the function which has value a at x and which vanishes at all other points. We then topologize $B_n(A,X)$ as the quotient

$$(A\times X)^n \to B_n(A,X), \qquad ((a_1,x_1),\dots,(a_n,x_n)) \mapsto (a_1)_{x_1} + \cdots + (a_n)_{x_n}.$$

Note that we have inclusions $B_n(A,X) \hookrightarrow B_{n+1}(A,X)$, and we define

$$A\otimes X := \operatorname*{colim}_n B_n(A,X),$$

with the topology of the union. Actually, McCord's construction can be described quite nicely as a tensor product of functors (see Nie).

Proof of the theorem. The usual way way to prove the Dold-Thom theorem is to show that the functor $X \mapsto \pi_*(\mathrm{Symm}^\infty(X,*)^+)$ satisfies the axioms for a reduced homology theory on based CW complexes, including the dimension axiom. The difficult axiom to check is that one obtains long exact sequences from cofiber sequences. To handle this, Dold and Thom introduced quasi-fibrations and proved a result allowing one to recognize quasi-fibrations.

We present instead a different proof, due to Spanier. For any simplicial set $K_\bullet$, let $\mathbb{N}K_\bullet$ denote the free simplicial commutative monoid on $K_\bullet$, which is just given by the free commutative monoid construction levelwise. Then there is a canonical *homeomorphism*

$$\mathrm{Symm}^\infty(|K_\bullet|_+) \cong |\mathbb{N}K_\bullet|$$

(here the $+$ denotes adding a disjoint basepoint). The essential point is that $\mathrm{Symm}^\infty(|K_\bullet|_+)$ is built out of colimits and finite products of copies of $|K_\bullet|_+$, and the geometric realization functor commutes with these constructions[1]. It is clear that the symmetric product construction on spaces preserves homotopies, and so the homotopy equivalence $X \simeq |S_\bullet(X)|$ for any CW complex X gives

$$\mathrm{Symm}^\infty(X,*) \simeq \mathrm{Symm}^\infty(|S_\bullet(X)|,*) \cong |\mathbb{N}(S_\bullet(X))/\mathbb{N}(*)|.$$

Finally, denoting $\mathbb{N}(K_\bullet)/\mathbb{N}(*)$ by $\tilde{\mathbb{N}}(K_\bullet)$ for a pointed simplicial set $K_\bullet$, it suffices to show that $\tilde{\mathbb{N}}K_\bullet \simeq \tilde{\mathbb{Z}}K_\bullet$ for any connected pointed $K_\bullet$ to complete the proof.

[1] Note that it is important that we work here in a good category of topological spaces, like compactly generated spaces, so that $|-|$ commutes with finite limits.

We will say something about the equivalence $\tilde{\mathbb{N}}K_\bullet \simeq \tilde{\mathbb{Z}}K_\bullet$ below, but note that if one is merely interested in producing a space whose homotopy groups are the homology groups of X, then that the above argument already gives $\mathbb{Z} \otimes X = \mathrm{Symm}^\infty(X,*)^+ \simeq |\tilde{\mathbb{Z}}(S_\bullet(X))|$, where here the plus denotes a naive group-completion.

We now address the equivalence $\tilde{\mathbb{N}}K_\bullet \simeq \tilde{\mathbb{Z}}K_\bullet$. By the Whitehead theorem, it suffices to show that $\tilde{\mathbb{N}}K_\bullet \hookrightarrow \tilde{\mathbb{Z}}K_\bullet$ is a H_*-isomorphism. Probably the sensible course at this point is to simply cite the spectral sequence argument of (Barratt-Priddy, 6.1).

Nevertheless, one can give a more explicit argument, at least in low degrees. First note that one can define a splitting s to $\mathbb{Z}\tilde{\mathbb{N}}K_\bullet \hookrightarrow \mathbb{Z}\tilde{\mathbb{Z}}K_\bullet$ by

$$s(\sum_i n_i\Big[\sum_j (m_{i,j} - m'_{i,j})\cdot k_{i,j}\Big]) = \sum_i n_i\Big[\sum_j m_{i,j}k_{i,j}\Big],$$

where $m_{i,j}, m'_{i,j} \geq 0$, for each i, j at most one of $m_{i,j}$ and $m'_{i,j}$ is nonzero, and the $k_{i,j}$'s are distinct. Since an element of $\mathbb{Z}\tilde{\mathbb{Z}}K_q$ is uniquely expressible in the above form, the section s is well defined. The existence of the section s gives that the maps $\mathrm{H}_*(\tilde{\mathbb{N}}K_\bullet) \hookrightarrow \mathrm{H}_*(\tilde{\mathbb{Z}}K_\bullet)$ is injective. It thus remains to show that we have a surjection in homology.

Since $K_\bullet$ is connected, so is $\tilde{\mathbb{Z}}K_\bullet$. Thus we have $\tilde{\mathrm{H}}_0(\tilde{\mathbb{N}}K_\bullet) \hookrightarrow \tilde{\mathrm{H}}_0(\tilde{\mathbb{Z}}K_\bullet) = 0$. It remains to obtain a surjection in higher degrees. The idea is to use the collapsed copy of $\mathbb{Z}$ at the basepoint, together with connectivity, to modify classes appearing with negative $\tilde{\mathbb{Z}}$ coefficients into the desired form.

We do this explicitly for $q = 1$. We will need two lemmas.

Lemma 1. *For any $v \in K_0$ and any cycle $\alpha = \sum_i n_i[\alpha_i]$ in $\mathbb{Z}\tilde{\mathbb{Z}}K_1$ we have $\sum_i n_i[s_0(v) + \alpha_i] \simeq \sum_i n_i[\alpha_i]$.*

Proof. One can check that if $\gamma_v \in \tilde{\mathbb{Z}}K_1$ is any edge connecting v to $* = 0$, then $\sum_i n_i([s_1(\gamma_v) + s_0(\alpha_i)] - [s_0(\gamma_v) + s_1(\alpha_i)])$ gives an explicit homotopy. The same argument works inducively for a chain of edges connecting v to $*$. □

Lemma 2. *For any $L \in \tilde{\mathbb{Z}}K_1$ and $I \in K_1$, we have*

$$[L - I + s_0d_0I + s_0d_1I] \simeq [s_0d_0L + I] - [s_0x_0 + L].$$

Proof. Here the homotopy is given by $[s_0s_0d_0I + s_1L - s_1I + s_0I]$. □

$$[s_0(e_0) + s_0(e_1) - x] \simeq [s_0(e_0)] - [x] \simeq -[x].$$

Now let $\sum_i n_i[\alpha_i] \in \mathbb{Z}\tilde{\mathbb{Z}}K_1$ be a cycle. Then if some α_i is $L - I$ with $L \in \tilde{\mathbb{Z}}K_1$ and $I \in K_1$, we can use the first lemma to replace $[\alpha_i]$ with $[\alpha_i + s_0d_0(I) + s_0d_1(I)]$. One can then use the second lemma to get rid of the negative coefficient in front of I. In this way, one can replace $\sum_i n_i[\alpha_i$ by a cycle in $\mathbb{Z}\tilde{\mathbb{N}}K_1$ as desired. ■

One useful consequence of the theorem is that it gives constructions of Eilenberg-Mac Lane spaces:

Corollary 1. $\mathrm{Symm}^\infty(S^n)$ *is a $K(\mathbb{Z}, n)$ $(n \geq 1)$.*

3. The equivariant version

Let G be a finite group. Note that the above constructions pass over to the equivariant setting. That is, if X is a G-space, then $\mathrm{Symm}^n(X)$ and $\mathrm{Symm}^\infty(X_+)$ naturally inherit G-actions. The question then becomes to identify the equivariant homotopy type of this space.

Theorem 2 (dos Santos)**.** *Let X be a based G-CW-complex, let V be a finite-dimensional G-representation, and let M be a $\mathbb{Z}[G]$-module. Then $M \otimes X$ is an equivariant infinite loop space and there is a natural isomorphism*

$$\pi_V^G(M \otimes X, *) \cong \tilde{\mathrm{H}}_V^G(X; \underline{M}).$$

Remark 4. Lima-Filho first proved the above result in the case $M = \mathbb{Z}$ (with trivial G-action) and where V is a trivial representation.

Before giving the proof, we recall some of the definitions involved in the statement of the theorem.

Definition 1. The **orbit category** $\mathscr{O}_G$ is the category with objects the orbits G/H and where $\mathscr{O}_G(G/H, G/K)$ is the set of maps of G-sets $G/H \to G/K$.

Definition 2. A **coefficient system** is a contravariant functor $\mathscr{O}_G^{op} \to Ab$.

Given a $\mathbb{Z}[G]$-module M, one can define a coefficient system $\underline{M}$ by $\underline{M}(G/H) = M^H$, the submodule of H-fixed points. Given any coefficient system N, there is a ($\mathbb{Z}$-graded) Bredon homology theory $\mathrm{H}_*^G(-; N)$ satisfying equivariant analogues of the Eilenberg-Steenrod axioms.

Finally, if M is a $\mathbb{Z}[G]$-module and X is a based G-space, we consider the construction $M \otimes X$ from the previous section as a G-space by taking the diagonal action of G.

Example 1. Note, however, that $(M \otimes X)^H \not\simeq M^H \otimes X^H$. For example, take $H = G = \mathbb{Z}/2\mathbb{Z}$, $M = \mathbb{Z}$ with trivial action, and $X = \mathbb{CP}^1$ with $\mathbb{Z}/2\mathbb{Z}$-action given by complex conjugation. One can show that $\mathbb{Z} \otimes \mathbb{CP}^1 \simeq \mathbb{CP}^\infty$ with G-action again given by complex conjugation. Thus $(\mathbb{Z} \otimes \mathbb{CP}^1)^{\mathbb{Z}/2\mathbb{Z}} \simeq \mathbb{RP}^\infty$. On the other hand, $\mathbb{Z}^{\mathbb{Z}/2\mathbb{Z}} \otimes (\mathbb{CP}^1)^{\mathbb{Z}/2\mathbb{Z}} \cong \mathbb{Z} \otimes \mathbb{RP}^1 \simeq S^1$.

Proof of the theorem. We only prove the theorem in the case $V = \mathbb{R}^n$ is a trivial representation.

We would like to mimic Spanier's proof from above. Note that if X is a G-CW complex, we have a G-homotopy equivalence $X \simeq |S_\bullet(X)|$, where here $|-| : G - sSet \rightleftarrows GTop : S_\bullet$ is the G-equivariant analogue of the usual adjoint pair. For instance, for a G-space X, the G-simplicial set $S_\bullet(X)$ is the usual singular complex on X, which then inherits a G-action from that on X.

Now the arguments from the nonequivariant proof still give

$$M \otimes X \simeq M \otimes |S_\bullet(X)| \cong |M \otimes S_\bullet(X)|,$$

but it is not clear how to directly identify the homotopy groups of the latter space with the Bredon homology groups of X. For a proof along these lines, see Nie.

Instead, we will show that $\pi_*(M \otimes X)$ satisfies the axioms for an ordinary equivariant homology theory. As in the nonequivariant case, the only axiom that needs work is exactness. We wish to show that $M \otimes (-)$ converts cofiber sequences into fiber sequences. By the above, it suffices to show that if (X, A) is a G-CW pair then the sequence

$$M \otimes S_\bullet(A) \to M \otimes S_\bullet(X) \to M \otimes \big(S_\bullet(X)/S_\bullet(A)\big)$$

is a G-fiber sequence in $G-sSet$. But since the induced map $\big(M \otimes S_\bullet(X)\big)/\big(M \otimes S_\bullet(A)\big) \to M \otimes \big(S_\bullet(X)/S_\bullet(A)\big)$ is an isomorphism of G-simplicial sets, one gets in particular that each

$$\big(M \otimes S_\bullet(X)\big)^H \to \Big(M \otimes \big(S_\bullet(X)/S_\bullet(A)\big)\Big)^H$$

is a surjective map of simplicial abelian groups and therefore a Kan fibration.

This shows that $\pi_*^G(M \otimes X)$ is an equivariant homology theory, and to finish the proof we check the dimension axiom. For this we will use a Lemma.

Lemma 3. *Let K be a based finite G-set. Then for any based G-space X there is an isomorphism of G-spaces*

$$\mathrm{Map}_*(K, M \otimes X) \cong M \otimes (K \wedge X),$$

where G acts on $Map_*(K, M \otimes X)$ by conjugation.

Now the computation is

$$\begin{aligned}\pi_n^G(M \otimes (G/H_+)) &\cong [S^n, M \otimes (G/H_+)]_G \cong [S^n, \mathrm{Map}_*(G/H_+, M)]_G \\ &\cong [S^n \wedge G/H_+, M]_G \cong [S^n, M]_H \cong \begin{cases} 0 & n \neq 0 \\ M^H & n = 0. \end{cases}\end{aligned}$$

This verifies the dimension axiom.

■