13. Lines On The Plane

There are three ways of looking at lines on the plane. One is geometric, one is algebraic, and one is physical. Geometrically, a line L is supposed to look as the x-axis. It is expressed by requiring that there is a parametrization $z(t)$, $t \in R$, such that the distance between $z(t)$ and $z(s)$ is exactly the same as the distance between t and s, i.e. $|z(t)-z(s)| = |t-s|$ for all $s, t \in R$. That way L has the same geometric properties as the x-axis.

Traditional way of defining lines algebraically is to represent them in the standard form, i.e. as solution set of $a \cdot x + b \cdot y + c = 0$ for some real numbers a, b, and c with $a^2 + b^2 > 0$.

Yet another way of looking at lines is more closely related to physics. It is the trajectory of a linear motion with constant velocity m. That means the formula for motion is $s(t) = m \cdot t + b$, where m is a non-zero complex number and b is a constant (the initial position in the motion). We assume that the time t varies from minus infinity to plus infinity.

The purpose of this section is to look at lines and segments from the point of view of complex numbers.

***Problem* 13.1.** (H)

Suppose a, b, c are real and both a and b are not zero at the same time. Show that $a \cdot x + b \cdot y + c = 0$ represents a line.

Hint(s) to 13.1: Can you solve for y? What if you cannot solve for y?

***Problem* 13.2.** (H)

Show that each line on the plane has equation of the form $a \cdot x + b \cdot y + c = 0$, where a, b, c are real and both a and b are not zero at the same time.

Hint(s) to 13.2: First consider vertical lines, then non-vertical ones.

***Problem* 13.3.** Show that each line on the plane has equation of the form $a \cdot z + b \cdot \bar{z} + c = 0$, where a is the conjugate of b, $b \neq 0$, and c is real.

***Problem* 13.4.** Suppose $a \cdot z + b \cdot \bar{z} + c = 0$ is an equation of a line on the plane so that c is real. Prove that a is the conjugate of b and $b \neq 0$.

***Problem* 13.5.** (A) The line passing through $1+2\cdot i$ and $4+5\cdot i$ has equation $a\cdot z+b\cdot\bar{z}+c=0$, where $c=2$. Find a.

Answer to 13.5: $1 + 1i$

***Problem* 13.6.** Let a, b, c be three complex numbers. Show that if c is the algebraic average of a and b (i.e., $c = (a + b)/2$), then c is the geometric midpoint between a and b (i.e., $|a-c| = |b-c| = |a - b|/2$).

***Problem* 13.7.** Let a, b, c be three complex numbers. Show that if c is the geometric midpoint between a and b (i.e., $|a - c| = |b - c| = |a - b|/2$), then it is the algebraic average of a and b (i.e., $c = (a + b)/2$).

***Problem* 13.8.** Suppose z and w are two different complex numbers and t is between 0 and 1. Prove that $v = t \cdot z + (1 - t) \cdot w$ lies on the segment between z and w.

***Problem* 13.9.** Suppose a, b, and c are three different complex numbers such that $|a-c|+|c-b|=|a-b|$. Prove that there is t between 0 and 1 so that $c=t\cdot a+(1-t)\cdot b$.

***Problem* 13.10.** Suppose z_1 and z_2 are two different points of plane. Prove that z lies on the line joining z_1 and z_2 if and only if there is a real number t such that $z = t \cdot z_1 + (1 - t) \cdot z_2$.

***Problem* 13.11.** Suppose z_1 and z_2 are two different points of plane. Prove that the line joining z_1 and z_2 contains z if and only if $(z - z_2)/(z_1 - z_2)$ is a real number.

***Problem* 13.12.** Suppose z_1 and z_2 are two different points of plane. Suppose z_3 and z_4 are two different points of plane. Prove that the line joining z_1 and z_2 is parallel to the line joining z_3 and z_4 if and only if $(z_1 - z_2)/(z_3 - z_4)$ is a real number.

***Problem* 13.13.** Suppose z_1 and z_2 are two different points of plane. Suppose z_3 and z_4 are two different points of plane. Prove that the line joining z_1 and z_2 is perpendicular to the line joining z_3 and z_4 if and only if $(z_1 - z_2)/(z_3 - z_4)$ is an imaginary number.

***Problem* 13.14.** Suppose z_1, z_2, and z_3 are three different points of plane. Prove that they form a right triangle at z_1 if and only if $|z_2 - z_3|^2 = |z_1 - z_2|^2 + |z_1 - z_3|^2$. Do not use Pythagoras Theorem. This is Pythagoras Theorem.

***Problem* 13.15.** Prove that the segment joining midpoints of two sides of a triangle is parallel to the third side and equal to half its length.

***Problem* 13.16.** Prove that any two medians of a triangle cut each other into segments whose lengths have ratio 2:1.

***Problem* 13.17.** Prove that the three medians of any triangle are concurrent.

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