Mathematics and Astronomy in India before 300 BCE

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1 Introduction

Our knowledge about the mathematical accomplishments in India prior to 300 BCE is derived primarily from ancient Sanskrit texts, especially the Vedic and the *Vedāṅga* treatises. The Vedic literature traditionally refers to the four *Saṁhita*s (*Ṛg, Sāma, Yajur* and *Atharva veda*s), *Brāhmaṇa*s, *Āranyaka*s, and *Upaniṣad*s. Though these are not technical texts, the number-vocabulary used in these treatises is of paramount mathematical significance.

In course of time, there arose the necessity of formalising some of the technical knowledge of the Vedic era. Thus emerged the six *Vedāṅga*s (literally, limbs of the Vedas): *śikṣā*(phonetics[1]), *chandas* (prosody), *vyākaraṇa* (grammar), *nirukta* (etymology), *jyotiṣa* (astronomy) and *kalpa* (rituals). Among the mathematical knowledge of the Vedic era that has been recorded in the *Vedāṅga*s, special mention may be made of (i) the significant geometrical results associated with the construction of the Vedic altars which have been presented in a portion of the *kalpa* known as the *Śulbasūtra*s,[2] (ii) certain rules on computation of metres in the *Chandaḥ-sūtra* of Piṅgalācārya,[3] and (iii) a formalised calendrical system with a five-year *yuga* and intercalary months that has been described in the *Vedāṅga-Jyotiṣa*.[4] They had a considerable influence on post-*Vedāṅga* literature.

That the study of mathematics was given an elevated status in India from at least the later Vedic Age, can be seen from certain passages of Upaniṣadic literature.[5] In an episode narrated in the *Chāndogya Upaniṣad* (7.1.2.4), the sage Sanatkumāra asks Nārada, a seeker of the supreme *Brahmavidyā*, to state the disciplines of knowledge

[1]The science of proper articulation and pronunciation.

[2]B. Datta (c), *The Science of the Śulba*; S.N. Sen and A.K. Bag, ed. with English translation and commentary, *Śulbasūtras of Baudhāyana, Āpastamba, Kātyāyana and Mānava.*

[3]Kedaranatha, ed. *Chandaḥ-sūtra of Piṅgala with the commentary Mṛtasañjīvanī of Halāyudha Bhaṭṭa.*

[4]T.S. Kuppanna Sastry and K.V. Sarma, *Vedāṅga Jyotiṣa of Lagadha.*

[5]See B. Datta and A.N. Singh, *History of Hindu Mathematics, Part I*, 3-4; B. Datta (d), "Vedic Mathematics" in P. Ray and S.N. Sen, eds., *The Cultural Heritage of India, Vol. VI*, 18.

he had already studied. In his list, Nārada explicitly mentions *nakṣatra-vidya* (the science of stars, i.e., astronomy) and *rāśi-vidya* (the science of numbers, i.e., mathematics). Such branches of *aparāvidyā*, i.e., worldly knowledge, were considered helpful adjuncts to *parāvidyā*, i.e., spiritual knowledge.

The importance of mathematics is again emphasised in the *Vedāṅga* literature. A verse in *Vedāṅga Jyotiṣa* asserts:[6]

> yathā śikhā mayūrāṇāṁ nāgānāṁ maṇayo yathā
> tadvadvedāṅgaśāstrāṇāṁ gaṇitaṁ mūrdhani sthitam
>
> As are the crests on the head of a peacock, as are the gems on the hoods of a snake, so is *gaṇita* (mathematics) at the top of the *śāstra*s known as the *Vedāṅga*.

The Jaina and Buddhist traditions too had a high esteem for the culture of mathematics.[7] One of the four branches of Jaina religious literature is *gaṇitānuyoga* (exposition of the principles of mathematics). A mastery over *saṁkhyāna* (the science of numbers, i.e., arithmetic) and *jyotiṣa* (astronomy) is stated to be one of the principal attainments of a Jaina priest.[8] The *Sthānāṅga-sūtra* considers mathematics to be *sukṣma* (subtle).[9] The *Sūtrākṛtāṅga-sūtra* (c. 300 BCE) regards geometry as the "lotus of mathematics".[10] As in Vedic tradition, mathematics and astronomy were relevant for the Jaina ceremonies. In Buddhist literature too, arithmetic, the science of numbers and calculations (called *saṁkhyāna*, *gaṇaṇa*), is ranked among the noblest of the arts.[11]

In this article, we shall highlight some of the significant mathematical and astronomical concepts that occur in Indian treatises prior to 300 BCE.[12] We give below a brief introduction to the contents of the different sections.

The striking feature of the Vedic number-vocabulary is that numbers are invariably expressed in the verbal form of our present decimal system. The invention of

[6] T.S. Kuppana Sastry and K.V. Sarma, op.cit., 36.

[7] See B. Datta and A.N. Singh, op.cit., 4, for precise references.

[8] See B. Datta (a), "The Jaina School of Mathematics", *Bulletin of the Calcutta Mathematical Society*, Vol. 21(2), 1929, 116.

[9] See B. Datta (a), op.cit., 123.

[10] See B. Datta (a), op.cit., 124, or B. Datta (b) , "The Scope and Development of Hindu Gaṇita", *Indian Historical Quarterly*, V, 1929, 491.

[11] Astronomy was not encouraged in Buddhist tradition, possibly because of its link with astrology. However, monks living in forests were advised to learn the stations of the constellations. They were to know the directions of the sky. See B. Datta (b), op.cit., 482.

[12] The much-admired town-planning and architectural proficiencies of the Harappan or Indus valley civilisation indicate the attainment of considerable sophistication in computational and geometric techniques. But, in the absence of textual evidence, we are not in a position to draw definite conclusions as to what exactly they knew about mathematics.

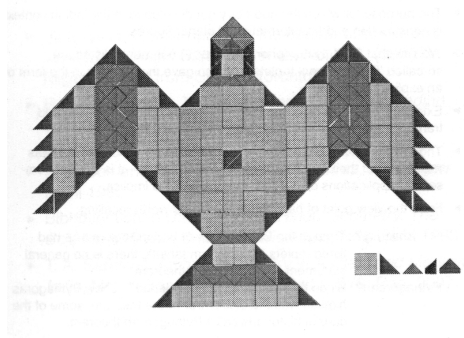


Figure 1: *śyenacit*.

this decimal system (even in its oral form) requires a high degree of mathematical sophistication.[13] Again, various references in the Vedic texts show that the fundamental operations of arithmetic were well-known at that time. We shall discuss these in Section 2.

The *Vedāṅga* that deals with rituals and ceremonies, namely the *Kalpa-sūtra*s, are broadly divided into two classes: the *Gṛhya-sūtra*s (rules for domestic ceremonies such as marriage, birth, etc.) and the *Śrauta-sūtra*s (rules for ceremonies ordained by the Veda such as the preservation of the sacred fires, performance of the *yajña*, etc.). The *Śulba-sūtra*s belong to the *Śrauta-sūtra*s.

Baudhāyana, Mānava, Āpastamba and Kātyāyana are the respective authors of four of the most mathematically significant *Śulba* texts, the Baudhāyana *Śulba-sūtra* being the most ancient. The language of the *Śulba-sūtra*s is regarded as being pre-Pāṇinian. The *Śulba-sūtra* of Baudhāyana (estimated to be around 800 BCE or earlier) is the world's oldest known mathematical text.

The *Śulba-sūtra*s give a compilation of principles in geometry that were used in designing the altars (called *vedi* or *citi*) where the Vedic sacrifices (*yajña*) were to be performed. The platforms of the altars were built with burnt bricks and mud mortar. The Vedic altars had rich symbolic significance and their designs were often intricate. For instance, the *śyenacit* in Figure 1 has the shape of a falcon in flight (a symbolic representation of the aspiration of the spiritual seeker soaring upward); the *kūrmacit* is shaped as a tortoise, with extended head and legs, the *rathacakracit* as a chariot-

[13] B. Bavare and P.P. Divakaran, "Genesis and Early Evolutiion of Decimal Enumeration: Evidence from Number Names in Ṛgveda", *Indian Journal of History of Science*, Vol. 48(4), 2013, 535-581; A.K. Dutta (e), "Was there Sophisticated Mathematics during Vedic Age?" in Arijit Chaudhury, et. al., eds., *An anthlogy of disparate technical thoughts at a popular level*, 91-132.

wheel with spokes, and so on. Further, the Vedic tradition demanded that these constructions are executed with perfection — the accuracy had to be meticulous — and the demand was met through remarkably sophisticated geometric innovations. We shall discuss a few of these geometric constructions and their underlying algebraic principles in Section 3.

In the last chapter of the prosody text *Chandaḥ-śūtra* of Piṅgalācārya (c. 300 BCE),[14] there are interesting mathematical rules embodied in *sūtra*s. The rules include the generation of all possible metres and the computation of their total number, corresponding to a given number of syllables. A crucial ingredient in these rules is a close analogue of the binary representation of numbers. The mathematical methods in the *Chandaḥ-sūtra* had a profound infuence on the development of combinatorial methods in later Indian texts. In Section 4, we discuss the enumeraive methods in *Chandaḥ-sūtra* and other texts.

References to astronomical phenomena associated with the motions of the Sun and the Moon like the solstices, equinoxes, solar year, seasons, lunar months, intercalary months, eclipses, and the *nakṣatra* system are to be found in the *Saṁhita*s and *Brāhmaṇa*s.[15] However, there is no presentation of a formal, quantitative system of astronomy in any of the extant literature of the pre-*Vedāṅga* period. The oldest available treatise exclusively devoted to astronomy is the *Vedāṅga Jyotiṣa* by Lagadha.[16] There are different views regarding the date of composition of this work, ranging from 1370 BCE to 500 BCE.[17] In this text, we see a definite calendrical system with a 5-year cycle of a *yuga*. In Section 5, we discuss astronomy in the earlier Vedic literature, *Vedāṅga Jyotiṣa* and also in subsequent works like *Arthaśastra* of Kauṭilya,[18] and some Jaina and Buddhist texts.[19]

In Section 6, which is the last one, we make a few concluding remarks.

[14]The date of this text is uncertain; tentative estimates vary from 500 to 200 BCE, with 300 BCE being the usually preferred date. Since Piṅgala has been referred to as an *anuja* of Pāṇini, a date closer to 500 BCE appears plausible.

[15]B. Datta (d), op.cit., 18-36; K.V. Sarma, "Indian Astronomy in the Vedic Age" in Siniruddha Dash, ed., '*Facets of Indian Astronomy*', 33-56; B.V. Subbarayappa, *Tradition of Astronomy in India - Jyotiḥśāstra*, 84.

[16]T.S. Kuppanna Sastry and K.V. Sarma, op.cit.

[17]T.S. Kuppana Sastry and K.V. Sarma, op.cit.; Y. Ohashi (a), "Development of Astronomical Observation on Vedic and Post-Vedic India", *Indian Journal of History of Science*, Vol. 28 (3), 1993, 185-251.

[18]R.P. Kangle (a), *The Kautilya Arthaśaāstra, Part I (text)*; R.P. Kangle (b), *The Kautilya Arthaśaāstra, Part II (translation)*.

[19]For a discussion on the dates of these texts, see S.N. Sen, "Survey of Source Materials" in D.M. Bose, S.N. Sen and B.V. Subarayappa, eds., *A Concise History of Science in India*, 34, 43.

2 Decimal system and Arithmetic in Vedic literature

The decimal system is a pillar of modern civilisation. It has been a major factor in the proletarisation of considerable scientific and technical knowledge, earlier restricted only to a gifted few. Due to its simplicity, children all over the world can now learn basic arithmetic at an early age. While the use of the perfected "decimal notation" (the written form of the decimal system) can be seen in written documents of the post-Vedic Common Era, the oral form of the decimal system goes back to the Vedic age. In this section, we shall discuss the Vedic number vocabulary that is based on the decimal system and that has been used in India throughout its subsequent history. We shall also give illustrations of arithmetical knowledge from Vedic literature.

For clarity, we first make a distinction between the two forms of the decimal system of representing numbers: the decimal notation and the decimal nomenclature. In our standard decimal notation, there is a symbol (called "digit" or "numeral") for each of the nine primary numbers (1,2,3,4,5,6,7,8,9), an additional tenth symbol "0" to denote the absence of any of the above nine digits, and every number is expressed through these ten figures using the "place-value" principle by which a digit d in the rth position (place) from the right is imparted the place-value $d \times 10^{r-1}$. For instance, in 1947, the symbol 1 acquires the place-value one thousand (1×10^3), 9 acquires the value nine hundered (9×10^2), etc. The Sanskrit word for "digit" is *aṅka* (literally, "mark") and the term for "place" is *sthāna*.

In the decimal nomenclature, each number is expressed through nine words ("one", "two", . . . ,"nine" in English) corresponding to the nine digits, and number-names for "powers of ten" ("hundred", "thousand", etc.) which play the role of the place-value principle. For convenience, some additional derived words are adopted (like "eleven" for "one and ten", . . ., "nineteen" for "nine and ten", "twenty" for "two ten", etc).

The verbal form of the decimal system was already in vogue when the *Ṛgveda* was compiled. Numbers are represented in decimal system in the *Ṛgveda*, in all other Vedic treatises, and in all subsequent Indian texts. The *Ṛgveda* contains the current Sanskrit single-word terms for the nine primary numbers: *eka* (1), *dvi* (2), *tri* (3), *catur* (4), *pañca* (5), *ṣaṭ* (6), *sapta* (7), *aṣṭa* (8) and *nava* (9); the first nine multiples of ten: *daśa* (10), *viṁśati* (20), *triṁśat* (30), *catvāriṁśat* (40), *pañcāśat* (50), *ṣaṣṭi* (60), *saptati* (70), *aśīti* (80) and *navati* (90), and the first four powers of ten: *daśa* (10), *śata* (10^2), *sahasra* (10^3) and *ayuta* (10^4). For compound numbers, the above names are combined as in our present verbal decimal terminology; e.g., "seven hundred and twenty" is expressed as *sapta śatāni viṁśatiḥ* in *Ṛgveda* (1.164.11).

An enunciation of the principle of "powers of ten" (a verbal manifestation of the abstract place-value principle), can be seen in the following verse of Medhātithi in the *Śukla Yajurveda* (verse 17.2 of the *Vājasaneyī Saṁhitā*), where numbers are being increased from one to one billion[20] by taking progressively higher powers of ten: *eka* (1), *daśa* (10), *śata* (10^2), *sahasra* (10^3), *ayuta* (10^4), *niyuta* (10^5), *prayuta* (10^6), *arbuda* (10^7), *nyarbuda* (10^8), *samudra* (10^9), *madhya* (10^{10}), *anta* (10^{11}), *parārdha* (10^{12}):

> imā me' agna' iṣṭakā dhenavaḥ santvekā ca daśa ca daśa ca śataṁ ca śataṁ
> ca sahasraṁ ca sahasraṁ cāyutaṁ cāyutaṁ ca niyutaṁ ca niyutaṁ
> ca prayutaṁ cārbudaṁ ca nyarbudaṁ ca samudraśca madhyaṁ cāntaśca
> parārdhaścaitā me' agna' iṣṭakā dhenavaḥ santvamutrāmuṣmilloke.[21]

Medhātithi's terms for powers of ten occur with some variations, sometimes with further extensions, in other *Saṁhitā* and *Brāhmaṇa* texts. Terms for much higher powers of ten are mentioned in subsequent Jaina and Buddhist texts and in the epic *Rāmāyaṇa*. When convenient, centesimal (multiples of 100) scales have been used in India for expressing numbers larger than thousand — the *Ṛgveda* (1.53.9) describes 60099 as *ṣaṣṭiṁ sahasrā navatiṁ nava* (sixty thousand ninety nine). The *Taittirīya Upaniṣad* (2.8) adopts a centesimal scale to describe different orders of bliss and mentions *Brahmānanda* (the bliss of Brahman) to be 100^{10} times a unit of human bliss; there is a similar reference in the *Bṛhadāraṇyaka Upaniṣad* (4.3.33).

In retrospect, a momentous step had been taken by ancient Vedic seers (or their unknown predecessors) when they imparted *single word-names* to successive powers of ten, thus sowing the seeds of the decimal "place-value principle".[22] The written

[20] Billion means 10^{12} (million million) in England and Germany but 10^9 (thousand million) in USA and France. Here we use it for 10^{12}.

[21] A literal translation could be: "O Agni! May these Bricks be my fostering Cows — (growing into) one and ten; ten and hundred; hundred and thousand; thousand and ten thousand; ten thousand and lakh; lakh and million; million and crore; crore and ten crores; ten crores and hundred crores; hundred crores and thousand crores; thousand crores and ten thousand crores; ten thousand crores and billion. May these Bricks be my fostering Cows in yonder world as in this world!". Note that in Vedic hymns, the Cow is the symbol of consciousness in the form of knowledge and the wealth of cows symbolic of the richness of mental illumination. The sanctified Bricks (*iṣṭakā*, i.e., that which helps attain the *iṣṭa*) are charged with, and represent, the mantras. See A.K.Dutta (b), "Powers of Ten: Genesis and Mystic Significance", *Srinvantu*, Vol. 48(2), 44-52.

[22] For expressing very large numbers in words, even the present English terminology (of using auxiliary bases like "thousand" and "million") is less satisfactory than the Sanskrit system of having a one-word term for each power of ten (up to some large power). This is effectively illustrated by

decimal notation is simply a suppression of the place-names (i.e., the single-word terms for powers of ten) from the verbal decimal expression of a number, along with the replacement of the words for the nine primary numbers by digits and the use of a zero-symbol wherever needed.

The decimal notation had evolved in India within the early centuries of the Common Era. It might have occurred even earlier. In the epic *Mahābhārata* (3.134.16), there is an incidental allusion to the decimal notation during the narration of a tale (3.132–134) involving ancient names like Uddālaka, Śvetaketu, Aṣṭāvakra, Janaka, et al, whose antiquity go back to the *Brāhmaṇa* phase of the Vedic era.[23]

Some Sanskrit scholars see in the term *lopa* (elision, disappearance, absence) of Pāṇini's grammar treatise *Aṣṭādhyāyī*, a concept analogous to zero as a marker for a non-occupied position, and have wondered whether *lopa* led to the idea of zero in mathematics, or the other way. Indeed, in a text *Jainendra Vyākaraṇa* of Pūjyapāda (c. 450 CE), the term "lopa" is replaced by *khaṃ*, a standard Sanskrit term for the mathematical zero. Pāṇini uses *lopa* as a tool similar to the null operator in higher mathematics. Unfortunately, mathematics texts of the time of Pāṇini have not survived.[24] In the prosody-text *Chandaḥ-sūtra*, Piṅgalācārya gives instructions involving the use of *dvi* (two) and *śūnya* (zero) as distinct labels. The choice of the labels suggests the prevalence of the mathematical zero and possibly a zero-symbol by his time.[25]

The decimal system (both in its verbal and notation forms) expresses any natural number as a polynomial-like sum $10^n a_n + \ldots + 100a_2 + 10a_1 + a_0$, where $a_0, a_1, \ldots, a_n$ are numbers between 0 and 9. Such a representation involves recursive applications of the well-known "division algorithm" that pervades the later Greek, Indian and modern mathematics. The mathematical sophistication of the decimal system can be glimpsed from the fact that its discovery required a realisation of the above principles.

The decimal system is largely responsible for the excellence attained by Indian mathematicians in the fields of arithmetic, algebra and astronomy. The dormant

G. Ifrah, *The Universal History of Numbers*, 428–429, by comparing the verbal representations of the number 523622198443682439 in English, Sanskrit and other systems.

[23]The precise phrase is *nava yogo gaṇanāmeti śaśvat*, "A combination of nine (digits) always (suffices) for any count (or calculation)." The word *śaśvat* (perpetual) has the nuance of "from immemorial time".

[24]The date of Pāṇini is uncertain; most estimates vary from 700 BCE to 500 BCE.

[25]The estimates of Piṅgalācārya's dates vary between 500 and 200 BCE; 300 BCE being the date used most frequently.

algebraic character of the decimal system influenced the algebraic thinking of mathematicians in ancient India and modern Europe. Post-Vedic ancient Indian geniuses like Brahmagupta (who defined the algebra of polynomials in 628 CE) and Mādhavācārya (who investigated the power series expansions of trigonometric functions in 14th century CE) had the advantage of being steeped in the decimal sytem gifted by the unknown visionaries of a remote past. As Isaac Newton would emphasise in 1671, the arithmetic of the decimal system provides a model for developing operations (addition, multiplication, root extraction, etc.) with algebraic expressions in variables (like polynomals and power series).[26] More recently, we see S.S. Abhyankar, a great algebraist of the 20th century, acknowledging the idea of decimal expansion in a technical innovation in his own research.[27]

Thanks to the decimal system, Indians developed efficient methods for the basic arithmetic operations which were slight variants of our present methods. The methods are described in post-Vedic treatises on mathematics but incidental references show that all the fundamental operations of arithmetic were performed during the Vedic time. For instance, in a certain metaphysical context, it is mentioned in *Śatapatha Brāhmaṇa* (3.3.1.13) that when a thousand is divided into three equal parts, there is a remainder one.[28] A remarkable allegory in the *Śatapatha Brāhmaṇa* (10.24.2.2-17) lists all the factors of 720:

$$720 \div 2 = 360; \quad 720 \div 3 = 240; \quad 720 \div 4 = 180; \quad 720 \div 5 = 144; \quad 720 \div 6 = 120;$$

$$720 \div 8 = 90; \quad 720 \div 9 = 80; \quad 720 \div 10 = 72; \quad 720 \div 12 = 60; \quad 720 \div 15 = 48;$$

$$720 \div 16 = 45; \quad 720 \div 18 = 40; \quad 720 \div 20 = 36; \quad 720 \div 24 = 30.$$

The *Pañcaviṁśa Brāhmaṇa* (18.3) describes a list of sacrificial gifts forming a geometrical series

$$12, 24, 48, 96, 192, \ldots, 49152, 98304, 196608, 393216.$$

The *Śatapatha Brāhmaṇa* (10.5.4.7) mentions, correctly, the sum of an arithmetical progression

$$3(24 + 28 + 32 + \ldots \ \text{ to } \ 7 \ \text{ terms } \ ) = 756.$$

[26]Newton's statement is quoted in P.P. Divakaran (a), "Notes on *Yuktibhāṣā*: Recursive Methods in Indian Mathematics" in C.S. Seshadri, ed., *Studies in the History of Indian Mathematics*, 296.

[27]See A.K. Dutta (e), op.cit., 105.

[28]The problem is also alluded to in the earlier *Ṛgveda* (6.69.8) and the *Taittirīya Saṁhitā* (3.2.11.2).

The *Bṛhaddevata*[29] gives the sum:[30]

$$2 + 3 + 4 + ... + 1000 = 500499.$$

The *sūtra*s 8.32-8.33 of *Chandaḥ-śūtra* of Piṅgala imply the following formula for the sum of the geometrical progression (G.P.):

$$1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1.$$

The following numerical example of the above G.P. series can be seen in the Jaina treatise *Kalpasūtra* (c. 300 BCE) of Bhadrabāhu:[31]

$$1 + 2 + 4 + ... + 8192 = 16383.$$

The incidental occurences of correct sums of such series in non-mathematical texts suggest that general formulae for series were known at least from the time of the *Brāhmaṇa*s.[32]

In the *Baudhāyana Śulba*, there are examples of operations with fractions like

$$7\frac{1}{2} \div (\frac{1}{5})^2 = 187\frac{1}{2};\ 7\frac{1}{2} \div (\frac{1}{15} \text{ of } \frac{1}{2}) = 225;\ \sqrt{7\frac{1}{9}} = 2\frac{2}{3};\ (3-\frac{1}{3})^2 + (\frac{1}{2} + \frac{10}{120})(1-\frac{1}{3}) = 7\frac{1}{2}$$

(Details and further examples from other *Śulba*s are given by B. Datta.[33])

Though the polynomial-type methods for performing arithmetical computations are described only in post-Vedic treatises, the polynomial aspect of the Vedic number-representation indicates that the Vedic methods too would have been akin to polynomial operations, using rules like (ten times ten is hundred), (ten times hundred is thousand), (hundred times hundred is ten-thousand) and so on, analogous to $xx = x^2, xx^2 = x^3, x^2x^2 = x^4$, etc.

The decimal system (both oral and written) enabled Indians to express large numbers effortlessly, right from the Vedic Age. This traditional facility with large numbers enabled Indians to work with large time-frames in astronomy (like cycles of 4320000 years) which helped them obtain strikingly accurate results.[34] Again,

[29] A treatise on Vedic deities ascribed to Śaunaka, a venerated Vedic seer. A.A. Macdonell places the text as being composed prior to 400 BCE.

[30] S.N Sen, "Mathematics" in D.M. Bose, S.N. Sen and B.V. Subbarayappa, eds., *Concise History of Science in India*, 144.

[31] S.N. Sen, op.cit.

[32] Explicit statements of the general formulae for the sum of A.P. and G.P. series occur later in the works of Āryabhaṭa (499 CE) and Mahāvīra (850 CE) respectively.

[33] B. Datta (c), *The Science of the Śulba*, Chapter XVI.

[34] For instance, Āryabhaṭa estimated that the Earth rotates around its axis in 23 hours 56 minutes and 4.1 seconds, which matches the modern estimate (23 hours 56 minutes 4.09 seconds).

it is due to the decimal system that post-Vedic Indian algebraists could venture into problems of finding *integer* solutions of linear and quadratic equations which often involve large numbers.[35] The traditional preoccupation with progressively large numbers, that was facilitated by the decimal system, created an environment that was conducive for the introduction of the infinite in Indian mathematics.[36]

The Vedic number system is the first known example of recursive construction. Recursive principles dominate Indian mathematical thought and are prominent features of some of its greatest achievements like the solutions of indeterminate equations and the work of the Kerala school.[37] The facility with recursive methods is another outcome of the decimal system.

A brief history of the decimal system is presented in the article by A.K. Dutta[38] and a detailed history in the source-book of Datta-Singh.[39]

3 Geometry and Geometric Algebra in Śulbasūtras

The *Śulbasutras* show insights on the geometric and algebraic aspects of the properties of triangles, squares, rectangles, parallelograms, trapezia and circles, and properties of similar figures. They describe geometric constructions for rectilinear figures (e.g., the perpendicular to a given line at a given point, a square on a given side, a rectangle with given sides, an isoceles trapezium with a given altitude, face and base) and exact methods for combination and transformation of geometric figures — forming a square by combining given squares or by taking the difference of two given unequal squares, transforming a rectangle (or an isoceles trapezium, an isoceles triangle, a rhombus) into a square and vice versa. In this section, we shall illustrate a few of their exact constructions, discuss their mathematical significance and high-

[35]For instance, the smallest pair of integers satisfying $61x^2 + 1 = y^2$ is $x = 226153980$, $y = 1766319049$. And this example occurs in the Algebra treatise *Bījagaṇita* (1150 CE) of Bhāskarācārya. (See A.K. Dutta (c), Kuṭṭaka, Bhāvanā and Cakravāla, in C.S. Seshadri ed., *Studies in the History of Indian Mathematics*, 145-199, for further details.)

[36]Indian algebraists like Āryabhaṭa and Brahmagupta (628 CE) had a mastery over indeterminate equations with infinitely many solutions, Bhāskarācārya (1150 CE) introduced an algebraic concept of infinity and also worked with the infinitesimal in the spirit of calculus, and then there was the spectacular work on infinite series by Mādhavācārya.

[37]A.K. Dutta (c), op.cit., 145-199; P.P.Divakaran (a), op.cit., 287-351; K.V. Sarma, K. Ramasubramanian, M.D. Srinivas and M.S. Sriram, *Gaṇita-Yukti-Bhāṣā of Jyeṣṭhadeva*.

[38]A.K. Dutta (d), op. cit., 1492-1504.

[39]B.Datta and A.N. Singh, *History of Hindu Mathematics, Part I*; Also see B. Bavare and P.P. Divakaran, "Genesis and Early Evolution of Decimal Enumeration: Evidence from Number Names in Ṛgveda"; B. Datta (d), *Vedic Mathematics*; G. Ifrah, op. cit.; Satya Prakash, *Founders of Sciences in Ancient India*; A.K. Dutta (e), op. cit., 91-132.

light the algebraic knowledge implicit in the *Śulba* methods. We first mention a theorem from the *Śulbasūtra*s which is a cornerstone of plane Euclidean geometry with applications throughout history: the celebrated result popularly known as the "Pythagoras Theorem".

The familiar version of Pythagoras Theorem states that the square of the hypotenuse of a right-angled triangle equals the sum of the squares of the other two sides. This result was known prior to Pythagoras (c. 540 BCE) in several ancient civilisations. However, the earliest *explicit statement* of the theorem occurs in the *Baudhāyana Śulba-sūtra* (1.48) in the following form (which we shall refer to as the "Baudhāyana-Pythagoras Theorem"):

> dīrghacaturaśrasyākṣṇayārajjuḥ pārśvamānī tiryaṅmānī ca yatpṛthagbhūte kurutastadubhayaṃ karoti.

Thus, Baudhāyana states that the square on the diagonal of a rectangle is equal (in area) to the sum of the squares on the two sides (which is clearly equivalent to the usual version of Pythagoras Theorem). The theorem is stated in almost identical language by Āpastamba (1.4) and Kātyāyana (2.11). In the Kātyāyana *Śulba*, there is an additional phrase *iti kṣetrajñānam* indicating the fundamental importance of the theorem in geometry. The theorem would play a pivotal role in much of ancient Indian geometry and trigonometry.

Though the Baudhāyana-Pythagoras Theorem is explicitly stated only in the *Śulba-sūtra*s of a late Vedic period, the result (along with other principles of Śulba geometry) was known and applied from the earlier phases of the Vedic era. The Baudhāyana-Pythagoras Theorem is a crucial requirement for the constructions of Vedic altars which are described in an enormously developed form in the *Śatapatha Brāhmaṇa* (a text much anterior to the *Śulba-sūtra*s); some of these altars are mentioned in the still earlier *Taittirīya Saṁhitā*. Further, the descriptions of the fire-altars in these older treatises are same as those found in the *Śulba-sūtra*s. In fact, the *Śulba* authors emphasise that they are merely stating facts already known to the authors of the *Brāhmaṇa*s and *Saṁhitā*s.[40] Even the *Ṛgveda Saṁhitā*, the oldest layer of the extant Vedic literature, mentions the sacrificial fire-altars (though without explicit descriptions of the constructions).

The *Śulba-sūtra*s are thus, in essence, engineering manuals for the construction

[40] See B. Datta (d), *The Science of the Śulba*, 25-40, and A. Seidenberg (b), 'The Origin of Mathematics', *Archive for History of Exact Sciences*, Vol. 18, 1978, 310-342.

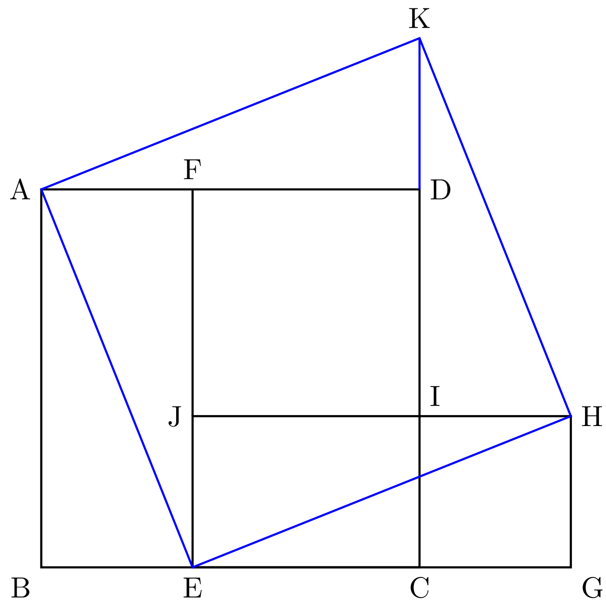


Figure 2: Baudhāyana's Construction of a square equal in area to the sum of two squares.

of fire-altars, summarising the necessary mathematical results and procedures which were already known over a long period of time. Detailed mathematical proofs or justifications are naturally outside the scope of these terse aphorismic handbooks. But, as emphasised by several scholars like Thibaut, Bürk, Hankel, Schopenhauer and Datta, the various *Śulba* constructions indicate that the *Śulba* authors knew *proofs* of the Baudhāyana-Pythagoras Theorem in some form.[41] As an illustration, we give below Baudhāyana's construction of a square equal in area to the sum of two given squares:

> nānācaturaśre samasyan kanīyasaḥ karaṇyā varṣīyaso vṛdhram ullikhet
> vṛdhrasya akṣṇayārajjuḥ samastayoḥ pārśvamānī bhavati

Thus, to combine the squares ABCD and ICGH as in Figure 2, the rectangle ABEF is cut off from the larger square ABCD such that its side BE has length equal to the side CG of the smaller square ICGH. Then the square AEHK on the diagonal AE of this rectangle ABEF is the required square. (That the area of AEHK is the sum of the areas of ABCD and ICGH can be seen by observing that the triangles ABE and EGH in the latter are being replaced, respectively, by the congruent triangles KHI and ADK.)

[41] B. Datta, op.cit., Chapter IX.

This construction, described (in verse 1.50) shortly after the statement of the Baudhāyana-Pythagoras Theorem (1.48), clearly shows that the Vedic savants knew why the Baudhāyana-Pythagoras Theorem holds. In fact, the diagram is itself a demonstration of the Baudhāyana-Pythagoras Theorem! For, it shows that the square AEHK on the diagonal AE of the rectangle ABEF is the sum of the square ABCD on the side AB and the square ICGH on the side CG(=BE). There are many more of such examples - two more will occur in this article in a different context.

A striking feature of the *Śulba* geometry is the abundance of "exact" constructions of the "straightedge-and-compass" type that makes our present high-school Euclidean geometry appear so formidable to a large section of students. These constructions demand mathematical rigour and do not allow measurements.[42] For instance, to obtain a square whose area equals the sum of the areas of the square ABCD and the square ICGH, one could have measured the length a of AB, the length b of IC, then mark out a side of length $\sqrt{a^2+b^2}$ (which will often be an irrational number even when a and b are rational numbers) and draw a square on it. But all these steps would have involved approximations. The purely geometric construction from Baudhāyana *Śulba* described above is free from any such measurement or approximation.

The mathematical sophistication of the Vedic age can be seen from the way the Baudhāyana-Pythagoras Theorem is applied to such geometric (exact) constructions in the *Śulba* treatises. The applications involve a subtle blend of geometric and algebraic thinking. An awareness of algebraic formulae like

$$(a \pm b)^2 = a^2 + b^2 \pm 2ab; \qquad a^2 - b^2 = (a+b)(a-b)$$

and quadratic equations, at least in a geometric form, is implicit in these constructions. The concern for accuracy in the building of the sacred fire-altars might have triggered the invention of the mathematical principles involved in these exact methods. We now present two more examples of exact constructions from the *Śulba* texts.

The Kātyāyana *Śulba-sūtra* (6.7) describes the following construction of a square equal in area to the sum of the areas of n squares of same size:

> yāvatpramāṇāni samacaturaśrāṇyekīkartuṁ cikīrṣedekonāni tāni bhavanti tiryak dviguṇānyekata ekādhikāni tryaśrirbhavati tasyeṣustatkaroti.

[42] Note that measurements inevitably involve inaccuracy. Due to the intrinsic inaccuracy, one takes several measurements during scientific experiments.

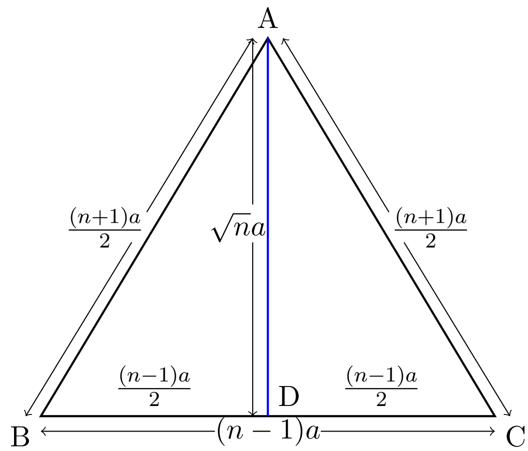


Figure 3: Kātyāyana's construction of a square whose area is n times the area of a smaller square.

Note that if each side of each of the given squares is of length a units, then the square to be constructed will have area na^2, i.e., each side of the desired square will be of length $\sqrt{n}a$. To construct a side of length $\sqrt{n}a$, the above verse prescribes constructing a line segment BC whose length is $(n-1)$ times the given length a and forming the isoceles triangle BAC with BC as the base such that each of the two sides BA and AC have length $\frac{(n+1)a}{2}$ (see Figure 3). Then the altitude DA of triangle BAC has the required length $\sqrt{n}a$ and the desired square can be constructed on this line segment DA. For, BD = $\frac{BC}{2} = \frac{(n-1)a}{2}$ and BA = $\frac{(n+1)a}{2}$, so that $DA^2 = (\frac{(n+1)a}{2})^2 - (\frac{(n-1)a}{2})^2 = na^2$.

Kātyāyana's procedure gives an *exact* construction of $\sqrt{n}a$ (no measurement or approximation is involved) making an ingenious application of the Baudhāyana-Pythagoras Theorem. It makes an implicit use of the formula

$$na^2 = (\frac{n+1}{2})^2 a^2 - (\frac{n-1}{2})^2 a^2;$$

in fact, the construction may be regarded as a geometric expression of the above algebra formula.

Next, we consider the Śulba procedure to construct a square equal in area to a given rectangle. Baudhāyana *Śulba* (1.54) states:

> dīrghacaturaśraṁ samacaturaśraṁ cikīrṣaṁstiryaṅmānīṁ karaṇīṁ kṛtvā śeṣaṁ dvedhā vibhajya viparyasyetaraccopadadhyāt khaṇḍamāvāpena tatsaṁpūrayet tasya nirhāraḥ utkaḥ.

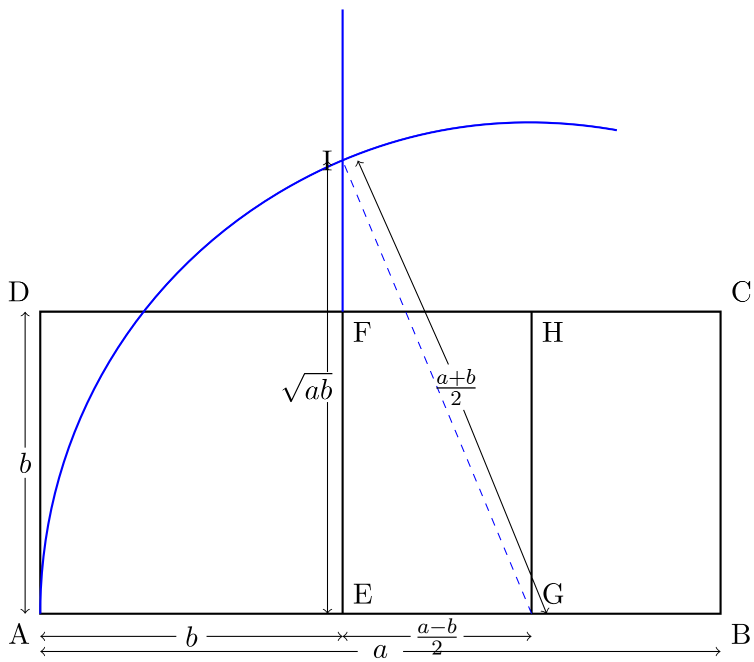


Figure 4: *Śulba* transformation of a rectangle into a square.

Thus, if ABCD is the given rectangle of length a units and breadth b units which is to be transformed into a square, then Baudhāyana prescribes marking out AE of length b units along AB (and complete the square DAEF) and bisecting the remainder EB (see Figure 4). With the mid-point G of EB as centre, an arc of radius AG is to be drawn which intersects the extension of the line EF at I. The segment EI gives the desired side. Note that $GE = \frac{a-b}{2}$ and $GI = GA = EA + GE = b + \frac{a-b}{2} = \frac{a+b}{2}$, so that $EI^2 = (\frac{a+b}{2})^2 - (\frac{a-b}{2})^2 = ab$.

Seidenberg remarks that this *Śulba* transformation of the rectangle into a square is in the spirit of Euclid's *Elements*:[43]

> "entirely in the spirit of *The Elements*, Book II, indeed, I would say it's more in the spirit of Book II than Book II itself."

The above procedure applies the Baudhāyana-Pythagoras Theorem to achieve an *exact* construction of $\sqrt{ab}$ from a and b through an implicit use of the formula

$$ab = (\frac{a+b}{2})^2 - (\frac{a-b}{2})^2;$$

it essentially gives a geometric formulation of the algebra formula.

[43] A. Seidenberg (b), op.cit., 318.

As will be clear from the above examples, the geometric knowledge in the Vedic era far transcended empirical observations — there was the mathematician's insight into the theorems and properties of geometric objects. Much of the mathematics of the Vedic savants was algebraic in spirit and substance and it is all the more remarkable that such accomplishments were made several centuries before the genesis of formal Algebra.[44]

The *Śulba* authors do not confine themselves to such exact constructions alone. An interesting statement in the *Śulba* verses of Baudhāyana (1.61–2), Āpastamba (1.6) and Kātyāyana (2.13) is the approximation:

$$\sqrt{2} \sim 1 + \frac{1}{3} + \frac{1}{3 \times 4} - \frac{1}{3 \times 4 \times 34}.$$

The right hand side equals the fraction $\frac{577}{408}$ which is the best possible approximation of $\sqrt{2}$ among fractions with the same or smaller denominators.[45] In terms of decimal fractions, $\frac{577}{408}$ ($= 1.4142156\ldots$) matches $\sqrt{2}$ ($= 1.414213\ldots$) up to five decimal places.[46] Simpler fractions $\frac{7}{5}$ and $\frac{17}{12}$ have also been used by *Śulba* authors as approximations for $\sqrt{2}$; they too have the property of being the most accurate among all fractions with denominators bounded by 5 and 12 respectively.

Another noteworthy feature of *Śulba* geometry is its study of the circle and formulation of rules to construct a circle from a square and vice versa. Such constructions are inevitably approximate.[47] The pioneering work on the circle in the Vedic era appears to have a significant impact on post-Vedic mathematicians and astronomers.[48]

In Section 2, we had mentioned examples of operations with fractions occurring in the *Śulba* treatises. The texts also show a familiarity with the addition, multiplication and rationalization of elementary surds; the term *karaṇī* was used for surd (e.g., *dvi-karaṇī* for $\sqrt{2}$). In Āpastamba *Śulba* (5.8), one sees an implicit use of results such as[49]

$$\frac{36}{\sqrt{3}} \times \frac{1}{2} \times (\frac{24}{\sqrt{3}} + \frac{30}{\sqrt{3}}) = 324; \qquad 12\sqrt{3} \times \frac{1}{2}(8\sqrt{3} + 10\sqrt{3}) = 324.$$

[44] The formalisation of Algebra was to occur more than a millennium later, possibly around the time of Brahmagupta (628 CE).

[45] A.K. Dutta (c), "Kuṭṭaka, Bhāvanā and Cakravāla", 186.

[46] The pair $(577, 408)$ satisfies the equation $x^2 - 2y^2 = 1$, a special case of an important equation investigated by Brahmagupta and other algebraists of the post-Vedic period; it is also involved in Ramanujan's prompt solution of a mathematical puzzle which had surprised P.C. Mahalanobis. See A.K. Dutta (f), "The *Bhāvanā* in Mathematics".

[47] Modern algebra has confirmed that it is not possible to make an exact construction of a square equal in area to a circle or vice versa, using straightedge-and-compass alone.

[48] One is reminded of Brahmagupta's brilliant results on quadrilaterals inscribed inside a circle.

[49] More details are to be found in B. Batta (c), *Science of the Śulba*, Chapter XVI.

The *Śulba-sūtra*s mention several rectangles the lengths of whose adjacent sides a, b and the length of each diagonal c are all integers (or rational numbers), i.e., (a, b, c) are integral (or rational) triples satisfying the famous equation $x^2 + y^2 = z^2$.[50] The triples $(3,4,5), (5,12,13), (7,24,25), (8,15,17), (12,35,37)$ and some of their multiples occur in the *Śulba-sūtra*s.[51]. The identity $4na^2 + (n-1)^2a^2 = (n+1)^2a^2$ that is implicit in Kātyāyana's rule for combining squares (discussed earlier in this section) suggests that Vedic scholars were aware that triples of the form $(2rs, r^2 - s^2, r^2 + s^2)$ satisfy $x^2 + y^2 = z^2$.

For constructing Vedic fire-altars, one has to find the numbers and sizes of different kinds of bricks required for building the different layers subject to various conditions. The altar-specifications amount to finding integer solutions of simultaneous indeterminate equations.[52] We find intricate examples of such specifications in Baudhāyana and Āpastamba *Śulba-sūtra*s.[53] Among the greatest mathematical achievements of post-Vedic stalwarts like Āryabhaṭa, Brahmagupta and Jayadeva are their systematic methods for finding integral solutions of linear and quadratic indeterminate equations.[54] This adds to the mathematico-historical significance of the implicit indeterminate equations in the *Śulba-sūtra*s.

Another interesting result of geometrico-algebraic flavour is Baudhāyana's method for constructing progressively larger squares, starting with a unit square, by adding successive gnomons.[55] This amounts to a geometric presentation (in fact, a geometric proof) of the algebraic identity

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

This is illustrated in Figure 5.

Due to the paucity of source-materials, we are not in a position to ascertain the full extent of the mathematical knowledge attained in the Vedic era.[56] But even

[50] An integer triple (a, b, c) satisfying $a^2 + b^2 = c^2$ is called a Pythagorean triple.

[51] See B. Datta (c), op.cit., 124, for more examples.

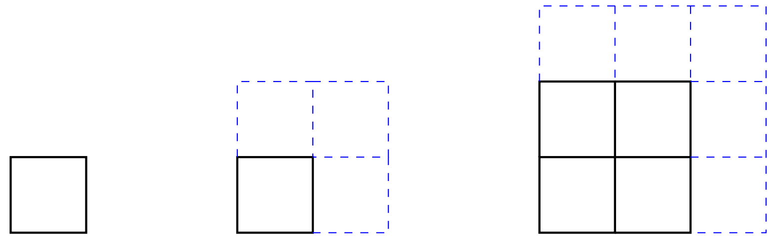
[52] A system of m algebraic equations in more than m variables is called *indeterminate*. The term is suggestive of the fact that such a system may have infinitely many solutions.

[53] For details, see A.K. Dutta (a), "Diophantine Equations: The Kuttaka", Resonance, Vol. 7(1), 2002, 8-10 and B. Datta (c), op.cit., Chapter XIV.

[54] For details see, A.K. Dutta (c), op.cit.

[55] Here, a gnomon refers to the L-shaped figure that one gets when a (smaller) square is removed from a corner of a square.

[56] Bibhutibhusan Datta mentions in B. Datta (b), "The scope and development of Hindu Gaṇita", 2, that apart from the practical geometry described in the *Śulba*s, the Vedic priests had also a secret knowledge of an esoteric geometry.



$1 = 1^2$ $\quad$ $1+3 = 2^2$ $\quad$ $1+3+5 = 3^2$

Figure 5: Demonstration of the result $1 + 3 + 5 + ... + (2n - 1) = n^2$ for $n = 1, 2, 3$.

what has come out on the basis of limited source-materials arouses a sense of sublime wonder among sincere scholars and thinkers on ancient Indian mathematics. The impact that contemplations on Vedic geometry can have on a dedicated seeker of its history can be seen from the tribute paid by Bibhutibhusan Datta (emulating a verse of Kālidāsa) in the Preface of his book:[57]

> "How great is the science which revealed itself in the *Śulba*, and how meagre is my intellect! I have aspired to cross the unconquerable ocean in a mere raft."

References for further reading. A systematic presentation of ancient Indian geometry, including Vedic geometry, occurs in the book of Sarasvati Amma.[58] The work of Bibhutibhusan Datta[59] contains a wealth of information and insights and is an indispensable beacon for anyone seriously interested in *Śulba* geometry. A. Seidenberg has made a masterly analysis of *Śulba* mathematics in his papers.[60] There are other papers which discuss specific features of *Śulba* mathematics.[61] Original texts of the *Śulba-sūtra*s with translations and commentaries are available in the literature.[62]

[57]B. Datta (c), op.cit.

[58]T.A. Sarasvati Amma, *Geometry in Ancient and Medieval India*.

[59]B. Datta (c), op.cit. Also see B. Datta and A.N. Singh (revised by K.S. Shukla), Hindu Geometry, *Indian Journal of History of Science*, Vol. 15(2), 1980, 121-188.

[60]A. Seidenberg (a), The Ritual Origin of Geometry, *Archive for History of Exact Sciences*, Vol. 1, 1962, 488-527; A. Seidenberg (b). op.cit.; A. Seidenberg (c), "The Geometry of Vedic Rituals", in F. Staal, ed., *Agni, The Vedic Ritual of the Fire Altar, Vol II.*

[61]D.W. Henderson, Square Roots in the Śulba Sūtras, in C.Gorini, ed., *Geometry at Work* ; J.F. Price, Applied Geometry of the Śulba Sūtras, in C. Gorini, ed., *Geometry at Work*; S.G. Dani, Geometry in the Śulvasūtras, in C.S. Seshadri, ed., *Studies in the History of Indian Mathematics*; P.P. Divakaran (b), "What is Indian about Indian Mathematics?", *Indian Journal of History of Science*, Vol. 51(1), 2016, 56-82; A.K. Dutta (e), "Was there sophisticated mathematics during Vedic Age?", in Arijit Chaudhuri, ed., *An anthology of disparate thoughts at a popular level*, 91-132.

[62]S.N. Sen and A.K. Bag, eds., *The Śulbasūtras of Baudhāyana, Āpastamba, Kātyāyana and Mānava*; Satya Prakash and R.S. Sharma, eds., *Baudhāyana Śulba Sūtra*; Satya Prakash and R.S. Sharma, eds, *Āpastamba Śulba Sūtra*; S.D. Khadilkar, ed., *Kātyāyana Śulba Sūtra*; G. Thibaut,

4 Enumerative mathematics in Piṅgala's *Chandaḥ-sūtra* and other works

4.1 Combinatorial mathematics in Piṅgala's *Chandaḥ-sūtra*

Piṅgala's *Chandaḥ-sūtra* (estimated around 300 BCE) systematises the rules for the 'metres' in Sanskrit poetry.[63] It has 31 *sūtra*s spread over 8 chapters. Of particular relevance to us are the mathematical concepts in the last 15 verses of the eighth chapter, where combinatorial tools and what amounts to a binary representation of numbers are used in the discussion of metrical patterns.[64]

The basic entities in the discussion are *laghu* (light) and *guru* (heavy) syllables, which we denote by L and G respectively. A syllable is *guru* if it has a long vowel or (even if is a short syllable), if what follows is a conjunct consonant, an *anusvāra*, or a *visarga*; otherwise it is a *laghu*. For instance, the sequence, "*sra-ṣṭu-rā-dhyā-va-ha-ti*" corresponds to G-L-G-G-L-L-L.

Prastāra

In the *sūtra*s (rules) 8.20-23, Piṅgala tells us how to obtain the *prastāra*s (layout with all the possible metrical patterns) for 1, 2 and 3 syllables:

1. Form a G, L pair.
2. Insert on the right, G's and L's.
3. [Repeating the process] we have eight (*vasavaḥ*) metrical forms in the 3-syllable *prastāra*.

The single syllable *prastāra* is :

1	G
2	L

In the 2-syllable *prastāra*, the first two rows are got by attaching a G at the right of each of the above, and the next two rows by attaching an L at the right of the above. So, we have the 2-syllable *prastāra*:

On the Śulva-sūtras, *Journal of the Asiatic Society of Bengal*, XLIV, 1875, 227-275; A.D. Bürk, "Das Āpastamba Śulva-sūtra", *Zeitschrift der deutschen morgenländischen Gesellschaft*, LV, 1901, 543-591 and LVI, 1902, 327-391.

[63] Kedaranatha, ed., *Chandaḥ-sūtra of Piṅgala with the commentary Mṛtasañjīvanī of Halāyudha Bhaṭṭa.*

[64] The articles, R. Sridharan (a),"Sanskrit Prosody, Piṅgala *Sūtra*s and Binary Arithmetic" in G.G. Emch, R. Sridharan and M.D. Srinivas, eds. "*Contributions to the History of Indian Mathematics*" and M.D Srinivas (a), "Piṅgala's *Chandaḥśāstra*", Lecture 5, NPTEL course on *Mathematics in India from Vedic Period to Modern Times*, www.nptel.ac.in, 2014, discuss the mathematical contents of Piṅgala's work. Our write-up follows the lecture note of M.D. Srinivas closely, and borrows extensively from it.

1	G	G
2	L	G
3	G	L
4	L	L

In the 3-syllable *prastāra*, the first four and the next four rows are obtained by attaching a G or an L respectively at the right of the 4 rows of the 2-syllable *prastāra*. So, we have the 3-syllable *prastāra*:

1	G	G	G
2	L	G	G
3	G	L	G
4	L	L	G
5	G	G	L
6	L	G	L
7	G	L	L
8	L	L	L

The procedure is likewise extended for metres with more syllables.[65]

The connection with the binary representation of numbers is the following. Set G $= 0$, and L $= 1$, and consider the mirror reflection of a row in any *prastāra*. Then that would be the binary representation of the row-number reduced by 1. Consider for instance, the 7th row in the 3-syllable *prastāra*: G L L. The mirror reflection is L L G $= 1\ \ 1\ \ 0 = 0 \times 1\ + 1 \times 2 + 1 \times 2^2 = 6$ in the binary representation, which is the row-number 7 reduced by 1.

Saṅkhyā

Now, the number of possible metres of n syllables is called *saṅkhyā*, which we denote by S_n. Its value is 2^n, as there are two possibilities for each syllable, namely L or G, so that for n syllables it is $2 \times 2 \times 2 \ldots n$ times, which is 2^n. *Sūtra*s 28-31 give an optimal algorithm for finding the number of metres with n syllables, i.e., 2^n:

a. Halve the number and mark "2".

b. If the number cannot be halved, deduct 1, and mark "0".

c. [Proceed till you reach 0. Start with 1 and scan the sequence of marks from the end].

d. If "0", multiply by 2.

e. If "2", square.

[65] In a later text called *Vṛttaratnāka* (c. 1000 CE), there is an alternate rule related to the above for generating the *prastāra*s, which gives the same results. See Madhusudana Sastri, ed., *Vṛttaratnākara of Kedāra with the commentaries, Nārāyaṇī and Setu.*

Example: Consider for instance, the casee of $n = 7$.

1. 7 cannot be divided by 2. 7-1 =6. Mark "0".

2. $\frac{6}{2} = 3$. Mark "2".

3. 3 cannot be halved. 3-1=2. Mark "0".

4. $\frac{2}{2} = 1$. Mark "2".

5. 1-1=0. Mark "0".

So, the sequence is 0 2 0 2 0. So beginning from the right, we have the sequence: $1 \times 2 = 2,\ 2^2,\ 2^2 \times 2 = 2^3,\ (2^3)^2 = 2^6,\ 2^6 \times 2 = 2^7$.

The same procedure is used for finding 2^n in all later texts on mathematics in India.

Sūtra 8.32 gives the sum of all the *saṅkhyā*s:

$$S_1 + S_2 + S_3 + \ldots + S_n = 2S_n - 2.$$

The next *sūtra* gives

$$S_{n+1} = 2S_n.$$

Together, the two *sūtra*s imply:

$$S_n = 2^n \text{ and } 1 + 2 + \ldots + 2^n = 2^{n+1} - 1.$$

The latter is clearly the formula for a geometric series. It is implicitly used for obtaining subsequent rules.

Naṣṭa, *Uddiṣṭa* and *Lagakriyā*

Suppose some rows in a *prastāra* are lost or "*naṣṭa*". They can be recovered using Piṅgala's procedure to find the metrical pattern corresponding to a given row. It is based on the binary representation of a number, and association of L with 1, and G with 0. The process (called *naṣṭa*) is stated in 8.24-25:

a. Start with the row number.

b. Halve it [if possible], and write an L.

c. If it cannot be halved, add 1 and halve, and write a G.

d. Proceed till all the syllables of the metre are found.

As an example, consider the construction of the 7th row of the 3-syllable *prastāra*.

1. 7 cannot be halved. $\frac{7+1}{2} = 4$, G.

2. $\frac{4}{2} = 2$, L.

3. $\frac{2}{2} = 1$, L.

So, the metrical pattern is G L L. Note that the mirror reflection of this is L L G, which corrresponds to $1\,1\,0 = 6$ in the binary representation. This is expected, as it is 1 less than the row number, which is 7.

Uddiṣṭa is the inverse of *naṣṭa*, that is, finding the row number given the metrical pattern. This is given in *sūtra*s 8. 26-27 of Piṅgala's *Chandaḥ-sūtra* and also in later texts like *Vṛttaratnākara*. We will not discuss it further, save mentioning that the rule is again associated with the binary representation of numbers, and the association of L with 1, and G with 0.

The *lagakriyā* process determines the number of metrical forms with r *guru*s (or *laghu*s) in a *prastāra* of metres of n syllables. This number is the binomial coefficient ${}^nC_r$ (the number of ways r objects can be chosen out of n objects).[66] Piṅgala's *sūtra* on this (8.34) is all too brief. Halāyudha, the tenth century commentator explains it as giving the basic rule for the construction of a table of numbers which he refers to as the *Meru-prastāra*.[67] Halāyudha's table is actually a rotated version of the well known Pascal triangle, and is based on the recurrence relation ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$.

In a recent article, Jayant Shah has claimed[68] that the *lagakriyā* is actually implied in some other *sūtra* of Piṅgala's *Chandaḥśāstra* in its *Yajur* rescension, namely (8.23b), and not the one cited above and elaborated by Halāyudha.

As the *laghu* and *guru* are associated with short and long syllables respectively, '*mātrā*' values of 1 for *laghu* and 2 for *guru* have also been assigned in the literature following Piṅgala's work. The *mātrā* value of any metrical form would be the sum of the *mātrā*s of each syllable. In the *Prākṛta* text *Vṛtta-jāti-samuccaya* (c. 600 CE),[69] Virahāṅka has discussed the problem of computing the total number of metrical forms M_n for a given value n of the *mātrā* and shown that $M_n = 1, 2, 3, 5, 8, 13, 21, \ldots$ for $n = 1, 2, 3, 4, 5, 6, 7, \ldots$, and satisfy the relation $M_n = M_{n-1} + M_{n-2}$.[70] This relation is satisfied by the so-called "Fibonacci numbers"[71] which are actually the Virahāṅka numbers M_n, whose discovery was inspired by Piṅgala's work.

[66] Mahāvīra (850 CE) and Herigone (1634 CE) gave the explicit formula ${}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r}$.

[67] Kedaranatha, ed., op.cit. Also see C.N. Srinivasiengar, *The History of Ancient Indian Mathematics*, 27-28; R. Sridharan (a), op.cit.; M.D.Srinivas (a), op.cit.

[68] Jayant Shah, "A History of Piṅgala's Combinatorics", *Gaṇita Bhāratī*, Vol. 35, 2013, 1-54.

[69] H.D. Velankar, ed., *Vṛtta-jāti-samuccaya or Kaisiṭṭha-chanda (in Prākṛta) of Virahāṅka, with commentary by Gopāla*.

[70] R. Sridharan (b), "*Pratyaya*s for *Mātrāvṛtta*s and Fibonacci Numbers", 120-137; M.D. Srinivas (b), "Development of Combinatorics", and references therein.

[71] described by Fibonacci around 1200 CE

4.2 Combinatorial, probabilistic and statistical ideas in other works

Vikalpa is the Jaina name for permutations and combinations. The Jaina text *Bhagavatīsūtra* (300 BCE) mentions the number of philosophical doctrines that can be formulated by combining a certain number n of basic doctrines, taking one at a time, two at a time, three at a time, four at a time, i.e., n $(= {}^nC_1)$, $\frac{n(n-1)}{1.2}$ $(= {}^nC_2)$, $\frac{n(n-1)(n-2)}{1.2.3}$ $(= {}^nC_3)$, $\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$ $(= {}^nC_4)$. The author further observes that "in this way, 5,7, ..., 10 etc., enumerable, unenumerable, or infinite number of things may be mentioned."[72] It is remarkable that apart from suggesting the number of combinations for a general n, mention is made of applying the method to infinite collections, and that there is a recognition that there are infinities of different cardinalities (i.e., sizes). This is indeed bold, considering the antiquity of the text.

A combinatorial statement also occurs in Chapter 26 of *Suśrutasaṁhita*, the celebrated medicinal work of Suśruta.[73] The text explicitly states (*sūtra* 4:12-13) that when the six different *rasa*s (tastes)[74] are taken one, two, ..., six at a time, there are $6(= {}^6C_1)$, $15(= {}^6C_2)$, $20(= {}^6C_3)$, $15(= {}^6C_4)$, $6(= {}^6C_5)$, and $1(= {}^6C_6)$ combinations and that their total is 63.[75]

In the *Mahābhārata* (*Vana parva*,72), there is an interesting episode in which King Ṛtuparṇa reveals to Nala the quick sampling method of estimating the number of leaves and fruits in a branch of a tree by counting them only on a portion of the branch.[76] P.C. Mahalanobis, who laid the foundations of statistics in modern India, observes that there are certain ideas in the Jaina logic *syādavāda* "which seems to have close relevance to the concepts of probability" and that *syādavāda* "seems to have given the logical background of statistical theory in a qualitative form".[77] Mahalanobis clarifies that it is not claimed that the concept of probability *in its*

[72]C.N. Srinivasiengar, op.cit., 26-27; A.K. Bag, *Mathematics in Ancient and Medieval India*, 188.

[73]M.S. Valiathan, *The Legacy of Suśruta*, 243; C.N. Srinivasiengar, op.cit., 27; A.K. Bag, op.cit., 188; M.D. Srinivas (b), "Development of Combinatorics in India 1", Lecture 18, NPTEL course on *Mathematics in India from Vedic Period to Modern Times*, www.nptel.ac.in, 2014. The range of dates for the text varies from 600 to 300 BCE; see R.C. Majumdar, "Medicine" in D.M. Bose, S.N. Sen and B.V. Subbarayappa, eds., *Concise History of Science in India*, 223-224.

[74]Bitter, sour, saltish, astringent, sweet and hot.

[75]C.N. Srinivasiengar, op.cit.; M.S. Valiathan, op.cit.; M.D. Srinivas (b), op.cit.

[76]This has been pointed out by the distinguished philosopher of science, Ian Hacking (*The Emergence of Probability: A Philosophical Study of Early Ideas about Probability, Induction and Statistical Inference*, 6-9) and by the eminent statistician, C.R. Rao (" Statistics in Ancient India", *Science Today*, December 1977, 36). For a more detailed discussion on the episode, see C.K. Raju, "Probability in Ancient India" in P.S. Bandyopadhyay and M.R. Forster, eds., *Handbook of the Philosophy of Science* Vol. 7, 2010, 1171–1192.

[77]P.C. Mahalanobis, The Foundations of Statistics, *Sankhyā*, Vol. 18, 1957, 183-194.;

present form was recognised in *syādavāda*, "but the phrases used in *syādavāda* seem to have a special significance in connection with the logic of statistical inference".

In the very first page of the Editorial of the inaugural issue of the journal *Sankhyā* (1933), P.C. Mahalanobis emphasises that "administrative statistics had reached a high state of organization before 300 B.C."[78] As he points out,[79] the *Arthaśāstra* of Kauṭilya "contains a detailed description for the conduct of agricultural, population, and economic censuses in villages as well as in cities and towns *on a scale which is rare in any country even at the present time.* ... The detailed description of contemporary industrial and commercial practice points to a highly developed statistical system." The *Arthaśāstra* also emphasises independent cross-verification of the collected data.[80]

5 Astronomy in India before 300 BCE

5.1 Astronomy in Vedic literature before *Vedāṅga Jyotiṣa*

There are references to astronomical phenomena and observations right from the early Vedic period.[81] The Sun was considered as the Lord of the universe who supports the heavens and the earth, controls the seasons and causes the winds. In the *Ṛgveda* it is stated that "God Varuṇa charted in the sky, a broad path for the Sun" (*ṚV* 1.24.8), which probably alludes to the zodiacal belt. In the *Taittirīya Saṁhita* (3.4.7.1), the Moon is referred to as '*Sūryaraśmi*', i.e., "(one who shines by) Sun's light". The dependence of Moon's phases on its elongation from the Sun is implicit in a description in *Śatapatha Brāhmaṇa* (1.6.4.18-20). This text also describes the earth explicitly as a sphere: *parimaṇḍala u va ayam lokaḥ* (7.1.1.37). The *Ṛgveda* (*ṚV*, 1.164.11) mentions the wheel of time formed with 12 spokes and 720 days and nights; the *Aitareya Brāhmana* (*AB*, 1.7.7) refers to a *saṁvatsara* with 360 days. It is possible that they are referring to a year with 360 days.

In early Vedic literature, we find references to both lunar and solar months. Now, a lunar month, being the time interval between two successive new Moons or full

[78] C.R. Rao too expresses the same view in C.R. Rao, op.cit.

[79] P.C. Mahalanobis, "Why Statistics?" *Sankhyā*, Vol. 10(3), 1950, 196-197.

[80] See P.C. Mahalanobis, op.cit., for the actual passages.

[81] B. Datta (d),"Vedic Mathematics", in P. Ray and S.N. Sen, eds., *The Cultural Heritage of India, Vol. VI*, 19-21; K.V. Sarma, "Indian Astronomy in the Vedic Age" in Siniruddha Dash, ed., *Facets of Indian Astronomy*, 33-56; S.N. Sen, "Astronomy" in D.M. Bose, S.N. Sen and B.V. Subbarayappa, eds., *Concise History of Science in India*, 58-135. All references to Vedic literature in this section are as in the article by K.V. Sarma, unless stated otherwise.

Moons, is a natural time-marker. Its average duration is nearly $29\frac{1}{2}$ days, i.e., there are 354 days in twelve lunar months, which is less than a year. From early times it was recognised that one needs to add '*adhikamāsa*'s or 'intercalary months' at regular intervals to align the lunar months with the solar year. The twelve months of the year (possibly solar months with 30 days each) have been named in the *Taittirīya saṃhita* which also gives the names of the intercalary months as '*saṃsarpa*' and '*aṃhaspati*' (*Taitt. Sam.* 1.4.14):

> "(O Soma juice!), you are taken in by the dish (*upayāma*). You are *Madhu, Mādhava, Śukra, Śuci, Nabhas, Nabhasya, Īṣa, Ūrja, Sahas, Sahasya, Tapas, Tapasya*. You are also *Saṁsarpa* and the *Aṁhaspati*."

The months are distributed among the six seasons. However, there is no mention of any rule regarding when the intercalary months are to be added. Seasons were determined by the position of the Moon. It has been suggested that the year being made of nearly 365 days is indicated in Vedic texts like *Taittirīya Saṁhita*.[82] Other possibilties of intercalation which would yield an average year of 365.25 days have also been suggested.[83] The basic concept of the calendar with 12 lunar months in a year with intercalary months at suitable intervals, is followed to this day in India, though in a more precise manner.

The northern and southern motions of the Sun (*uttarāyana* and *dakṣiṇāyana*) are referred to in *Ṛg, Yajur* and *Atharva veda*s. The equinoxes at the middle of the *ayana*s and the solstices at their beginning are mentioned. It is noted that the Sun stands still at the winter and summer solstices.[84] A 5-year *yuga*-cycle is also mentioned in *Taittirīya* and *Vājasaneyi saṁhita*s.

As the sidereal period of the Moon is close to 27.11 days, i.e., the Moon covers nearly $\frac{1}{27}$th part of the ecliptic[85] per day (angle-wise), it is natural to divide the

[82] In his introduction to the work on *Vedāṅga Jyotiṣa*, T.S. Kuppanna Sastry observes: "The solar year was known to have 365 days and a fraction more, though it was roughly spoken of as having 360 days, consisiting of 12 months of 30 days of each. Evidence of this is found in the *Kṛṣna-yajurveda: Taittirīya Saṁhita* (TS) 7.2.6, where the extra 11 days over the 12 lunar months, totalling 354 days, is mentioned to complete the *ṛtu*s by the performance of the *Ekādaśa rātra* or eleven-day sacrifice. TS 7.1.10 says that 5 days more were required over the *Sāvana* year of 360 days to complete the seasons, addding that 4 days are too short and 6 days too long." T.S. Kuppanna Sastry and K.V. Sarma, *Vedāṅga Jyotiṣa of Lagadha*, 10. The relevant passages from TS are in page 20.

[83] S.N. Sen, op.cit., 75-76.

[84] This is due to the fact that the declination of of the Sun has the least variation at these points, due to which the points of rising and settting of the Sun on the horizon do not vary over many days.

[85] Ecliptic is the circle which is the apparent path of the Sun around the earth in the background of stars. It is inclined at an angle of nearly $23\frac{1}{2}^{\circ}$ with the celestial equator.

ecliptic into 27 equal divisions. Each of these divisions is called a '*nakṣatra*', so that each day is associated with a *nakṣatra* in which the Moon is situated. *Aśvini, Bharaṇi, Kṛittikā, Rohiṇi, Mṛgaśira, Ārdrā, Punarvasu, Puṣya, Āśleṣa, Maghā, Pūrva Phālguṇi, Uttara Phālguṇi, Hasta, Citrā, Svāti, Viśākhā, Anurādhā, Jyṣṭhā, Mūlā, Pūrvāṣāḍhā, Uttaāṣāḍhā, Śravaṇa, Dhaniṣthā, Śatabhiṣaj, Pūrvābhādrā, Uttarābhādrā, and Revati* are the 27 *nakṣatra*s. The full list of 27 *nakṣatra*s headed by *Kṛittikā* appears in *Taittirīya saṁhita* and *Atharvaveda*.[86]

B.V. Subbarayappa has pointed out that the nomenclature of some of the *nakṣatra*s had agricultural significance:[87]

> "The word *Ārdrā* means 'wet' and the *nakṣatra Ārdrā* heralded the onset of rains when the Sun became positioned in it ... *Puṣya* denoted the growth and nourishment of young sprouts ... *Maghā* meant the wealth of standing fruitful crop."

In *Ṛg* and *Atharva veda*s, five celestial objects are mentioned, as being distinct from the stars. These are the planets Mercury, Venus, Mars, Mars, Jupiter and Saturn. Jupiter and Venus are mentioned by name.[88] Allusions to eclipses can be seen in verses mentioning darkness hiding the Sun (solar eclipse) and the Moon entering the Sun (lunar eclipse). The descendants of the sage Atri are said to be knowledgeable about the eclipses.[89]

5.2 *Vedāṅga Jyotiṣa* and allied literature

We have seen in early Vedic literature the rudiments of a calendar with intercalary months added to ensure that the lunar months are in step with the seasons, and with 27 *nakṣatra*s as markers of Moon's movement. However, all descriptions there are qualitative. It is in *Vedāṅga Jyotiṣa* that we have a definite quantitative calendrical system.[90] One of the limbs of the Vedas, this work is attributed to sage Lagadha and comes in two rescensions: the *Ṛgvedic* (36 verses), and the *Yajurvedic* (43 verses);

[86] S.N. Sen, op.cit., 66-68; B.V. Subbarayappa and K.V. Sarma, *Indian Astronomy: A Source Book*, 110-111. The Babylonians had a series of 33 or 36 zodiacal stars. Also there were *hsui*s or stars associated with the lunar zodiac stars in Chinese records. However, there is no evidence of any influence of these on the Indian *nakṣatras* (S.N. Sen, op.cit., 79-82.), and the "indigenous origin of the *nakṣatra* system can never be in doubt" (B.V. Subbarayappa, *The Tradition of Astronomy in India, Jyotiḥśāstra*, 84).

[87] B.V. Subbarayappa, op.cit., 84.

[88] K.V. Sarma, op. cit., 53-54.

[89] K.V. Sarma, op.cit., 51-54; S.N Sen, op.cit., 64.

[90] T.S. Kuppanna Sastry and K.V. Sarma, op.cit.

their basic content is the same. There is reference to the winter solstice being at the beginning of the asterism *Śraviṣṭhā* (Delfini) segment, and the summer solstice at the mid-point of the *Āsleṣa* segment. This would corrrespond to some time between 1370 BCE and 1150 BCE, taking into account the precession of the equinoxes and possible errors in the precise locations of the solstices.[91]

The calendrical system of *Vedāṅga Jyotiṣa* is as follows. A *yuga* has 5 solar years each consisting of 366 civil days, i.e., there are 1830 civil days in a *yuga*. When the Sun and the Moon are at the beginning of the winter solstice (in *Śraviṣṭhā*, as we saw), it is the beginning of the *yuga*, the first solar year, the first lunar month, and the first *ayana* (*uttarāyana* in this case). Each year has 2 *ayana*s and 6 seasons. After the completion of one *yuga*, the Sun and the Moon come together at the same position in the stellar background. There are 67 sidereal lunar months (the time required by the Moon to complete one revolution), and 62 (=67 - 5) synodic or lunar months in a *yuga*. Each lunar month has two *pakṣa*s or *parva*s (bright and dark), and each of them has 15 *tithi*s, so that there are 124 *parva*s. *Tithi* is a concept which is unique to India, and is explicitly mentioned for the first time in *Vedāṅga Jyotiṣa*.[92]

As there are 62 lunar months in 5 years, the *Vedāṅga Jyotiṣa* adds two additional or 'intercalary months' (*adhikamāsa*s) in each cycle of 5 years: one each in the third year and the fifth year.

The *Vedāṅga Jyotiṣa* calendar is based on the 'mean' or average motions of the Sun and the Moon. In later times, the Indian calendar retains the concepts of intercalary months, *tithi*s, *nakṣatra*s, etc., but all the calculations are based on the true motions of the Sun and the Moon, which are not uniform.

There are arithmetical rules regarding the occurrence of various phenomena associated with the Sun and the Moon. Two important reference points in astronomy are the points of intersection of the ecliptic and the equator, called equinoxes (*viṣuvat* in Sanskrit). They are the mid-points of the *uttarāyana* (northward motion), or the *dakṣināyana* (southward motion) of the Sun. Verse 31 in *Ṛg* rescension and verse 23 in the *Yajur* rescension of the *Vedāṅga Jyotiṣa* tell us how to calculate the instant

[91] However, the text itself could have been composed later, but before 500 BCE. See T.S.Kuppana Sastry and K.V. Sarma, op.cit. and Y. Ohashi (a), "Development of Astronomical Observation on Vedic and Post-Vedic India", *Indian Journal of History of Science*, Vol. 28 (3), 1993, 185-251.

[92] During each *tithi*, the angular separation between the Sun and the Moon increases by 12°.

of the nth equinox in terms of the number of *parva*s and *tithi*s:[93]

> "Take the ordinal number of the *viṣuvat* and multiply by 2. Subtract one. Multiply by 6. What has been obtained are the number of *parva*s gone. Half of this is the *tithi* at the end of which the *viṣuva* occurs."

This can be understood as follows. The Sun traverses the ecliptic 5 times in a *yuga*, and the interval between two *viṣuvat*s corespond to half of the ecliptic. Also there are 124 *parva*s in a *yuga*. Hence the interval between two successive *viṣuvat*s is $\frac{124}{10}$ *parva*s = 12 *parva*s, 6 *tithi*s (as 1 *parva* = 15 *tithi*s). Hence, the instant of occurrence of the n^{th} *viṣuvat* is $(n-\frac{1}{2})(12\,parvas\;\;6\,tithis) = (2n-1)\times 6\,parvas +\;(2n-1)\times 3\,tithis$, which is the rule. There are more complex arithmetical rules regarding the instants at which a lunar or solar *nakṣatra* begins, and other instants.

The calendrical system of *Vedāṅga-Jyotiṣa* is followed in many later texts, like *Arthaśāstra* of Kauṭilya (around the fourth century BCE), *Śārdūlakarṇāvadāna* (a Buddhist text around the third century BCE, which was translated into Chinese in third century CE), several Jaina Prākrit texts like *Sūrya-prajñāpti* and *Candra-prajñāpti* (around 300 BCE), and *Paitāmaha-siddhānta* of first century CE.[94]

Duration of day-time

The duration of the day-time (time-interval between the Sunrise and the Sunset) varies over the year, depending upon the position of the Sun on the ecliptic (specifically, its declination which is its angular separation from the equator), and also the latitude of the place. On the equinoctial day, when the Sun is on the equator, the durations of the day-time and the night-time are both equal to 15 *muhūrta*s for all the latitudes.[95] At the winter solstice, the day-time is the least and at the summer solstice, it is the maximum. *Vedāṅga-jyotiṣa* gives a simple arithmetical rule for the duration of the day-time over the year in verse 22 of *Ṛg* rescension and verse 40 of *Yajur* rescension:[96]

> "The number of days which have elapsed in the northward course of the Sun (*uttarāyana*) or the remaining days in the southward course (*dakṣināyana*) doubled and divided by 61, plus 12, is the day-time (in *muhūrta*s) of the day taken."

[93] T.S. Kuppanna Sastry and K.V. Sarma, op.cit., 47.
[94] Y. Ohashi (a), op.cit.
[95] A *muhūrta* is one-thirtieth of a civil day.
[96] T.S. Kuppanna Sastry and K.V. Sarma, op.cit., 66.

Hence, the duration of day-time is given by

$$D_t = (12 + \frac{2n}{61})\ \ muh\bar{u}rtas,$$

where n denotes the number of days elapsed after the winter solstice when the Sun's course is northward, and the number of days yet to elapse before the winter solstice when the Sun's course is southward. On the winter solstice day, $n = 0$, and $D_t = 12$ *muhūrta*s; at the equinox (*viṣuvat*), $n = 91.5$, and $D_t = 15$ *muhūrta*s; and at the summer solstice, $n = 183$, and $D_t = 18$ *muhūrta*s.

Actually, D_t depends upon the latitude of the place, and the text does not specify the place where the formula is valid. The ratio of the daytimes for the summer and winter solstices is $18 : 12 = 3 : 2$ according to the formula, and this is true for a latitude of 35°N using the modern formula for the day-time.[97] However, the values of D_t using the modern formula for this latitude do not agree with the *Vedāṅga Jyotiṣa* rule for most days. Ohashi showed that the rule works well for latitudes between 27° and 29° N, for most days and is probably based on observations.[98] The following figure depicts the variation of the day-time, D_t with the number of days elapsed after the winter solstice.[99] As the longitude of the Sun, λ varies uniformly in *Vedāṅga-Jyotiṣa*:

$$\lambda = -90^\circ + \frac{n}{183} \times 180^\circ,$$

where n denotes the number of days elapsed after the winter solstice. Here, $\lambda = -90^\circ$ at the winter solstice ($n = 0$), and $\lambda = 90^\circ$ at the summer solstice ($n = 183$). The duration of the day-time for partcular values of λ and the latitudes of the place is easily calculated using modern spherical astronomy.[100] We compare the variation of the day-time with n using the *Vedāṅga-Jyotiṣa* formula (straightline in the figure), and the modern formula for latitudes 27°, 29° and 35°N. For the first two latitudes, there is remarkable agreement with the rule in the text, except near the solstices, as pointed out earlier, whereas for the latitude of 35°N, the agreement with the rule is good only at the solstices and the equinox.

The use of gnomon for determining directions

[97] See for instance, W.M. Smart, *Textbbook on Spherical Astronomy.*

[98] Y. Ohashi (a), op.cit., 205; Y. Ohashi (b), "Mesopotamian Zig-zag Function of Day Length from Indian Point of View", *Gaṇita Bhāratī*, Vol. 34, 2012, 53-64.

[99] This figure is in the same spirit as the ones in the papers by Ohashi, where the duration of day-time is plotted against Sun's lomgitude, λ in the interrval 0° to 90° for 27°N and 29°N longitudes, and compared with the values in the text.

[100] W.M. Smart, op.cit.

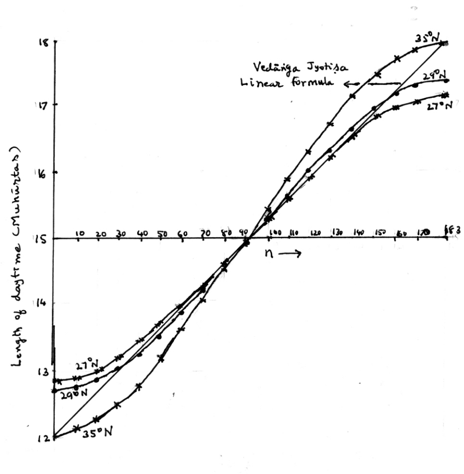


Figure 6: Comparison of the length of day-time calculated from the modern formula for latitudes 27°, 29° and 35°N, with the *Vedāṅga Jyotiṣa* rule.

A prominent feature of the *śulba* texts is the reference to the west-to-east direction (*prācī*, the eastward line). Every Vedic fire-altar has a principal line of symmetry which is to be placed along this direction. All geometric constructions in the *Śulba* texts are described with reference to this west-east line and its perpendicular (the north-south line). The fixing of these cardinal directions is one of the features of *Śulba* geometry, which seems to have influenced post-Vedic trigonometry with its emphasis on the sine function (as in modern trigonometry).[101] The *Kātyāyana Śulba-sūtra* (I.2) describes the determination of the east-west line:[102]

> "Having put a gnomon (*śaṅku*) on a level ground, and having described a circle with a cord whose length is equal to the gnomon, two pins are placed on each of the two points where the tip of the gnomon-shadow touches [the circle in the forenoon and afternoon respectively]. This [line joining the two points] is the east-west line (*prācī*)".

Annual and the diurnal variations of the shadow

In *Artha-śāstra* (II.20.41-42) it is stated that the mid-day shadow of a 12-digit gnomon is zero at the summer solstice, and increases at the rate of 2 digits per month

[101] See P.P. Divakaran (b), "What is Indian about Indian Mathematics?", *Indian Journal of History of Science*, Vol. 51(1), 2016, 67-68.

[102] S.N. Sen and A.K. Bag, ed. with English translation and commentary, "*Śulbasūtras*", New Delhi, 1983; Y. Ohashi (a), op. cit.

during the Sun's southward course towards the winter solstice.[103] The Buddhist text *Śardūlakarṇāvadāna*, and the Jaina texts also give lists of shadow-lengths every month.

The *Artha-śāstra* (II.20.39-40) and the Jaina text *Candra-prajñāpti* give similar data for the diurnal variation of the shadow.[104] The *Artha-śāstra* values fit very well with the actual shadow at the summer solstice for a latitude of 23.7°N. The *Śardūlakarṇāvadāna* and the *Atharva-Jyotiṣa* data on the diurnal variation of the shadow fit reasonably with the actual shadows at the equinox. From the stated numbers in the various texts on the annual and diurnal variations of the shadow, Ohashi concludes that they are based on actual observations in north India.[105]

6 Concluding remarks

We have seen that some of the major mathematical and astronomical concepts in India can be traced to Vedic times. The ideas are elaborated in the *Vedāṅga* and *Sūtra* period; there were contributions from the "heterodox" Jaina and Buddhist schools too. This corpus of literature before 300 BCE had a profound impact on the development of mathematics and astronomy in India in the later period.[106] We have already remarked on the centrality of the decimal system for the excellence attained in Indian arithmetic, algebra and astronomy. The polynomial-type methods for performing algebraic operations are similar to operations involving numbers in the decimal place value system. Again, it is due to the decimal system that Indians in the classical age could attempt the problems of finding *integer* solutions of linear and quadratic indeterminate equations which often involve very large numbers, and develop their mathematical astronomy which again involved large numbers.

Śulba geometry based on the Baudhayana-Pythagoras Theorem played a pivotal role in the development of geometry, trigonometry and astronomy in the later period. The emergence of calculus concepts in India in the form of infinite series for π and

[103] R.P. Kangle (a), op.cit., 71; R.P. Kangle (b), op.cit., 139; Y. Ohashi (a), op.cit., 208-209.

[104] Y. Ohashi (a), op.cit., 214-217.

[105] See Y. Ohashi (a), op.cit., 214, 225.

[106] B. Datta and A.N. Singh, *History of Hindu Mathematics, Parts I and II* ; B. Datta, A.N. Singh (revised by K.S. Shukla), "Hindu Trigonometry", *Indian Journal of History of Science*, Vol. 18, 1983, 39-108; C.N. Srinivasiengar, *The History of Ancient Indian Mathematics*; A.K. Bag, *Mathematics in Ancient and Medieval India*; T.A. Saraswati Amma, *Geometry in Ancient and Medieval India*; Kim Plofker, *History of Mathematics in India: From 500 BCE to 1800 CE*; D.A. Somayaji, *A Critical Study of Ancient Hindu Astronomy*; S.N. Sen and K.S. Shukla, eds, *A History of Indian Astronomy*; B.V. Subbarayappa and K.V. Sarma, op.cit.; B.V. Subbarayappa, op.cit.

trigonometric functions and integration methods[107] was facilitated by the decimal system, and the geometrico-algebraic methods of the *Śulba-sūtra*s.

Combinatorial ideas in Piṅgala's *Chandaḥ-sūtra* and other texts are elaborated and further developed in later works. We have seen in Section 4.1 that Virahāṅka's discovery (around 600 CE) of the so-called Fibonacci numbers was inspired by Piṅgala's work (around 300 BCE). Again, while the post-Vedic *Gaṇitasārasaṅgraha* (850 CE) of Mahāvīra gives the general formula for ${}^nC_r$, we have seen that Piṅgala's work gives a hint for finding ${}^nC_r$, and explicit values of ${}^nC_r$'s are mentioned for specific values of n in various pre-300 BCE texts. Further progress is recorded in the works of Bhāskarācārya (12th century), Nārāyaṇa Paṇdita (14th century) and others.[108] We also see sophisticated combinatorics in the thoery of Indian music, where the *pratyaya*s (procedures) are essentially the same as in Piṅgala's seminal work.[109]

We cite one example of the influence of *Vedāṅga-Jyotiśa* in post-Vedic *Siddhānt*ic astronomy. In the 5-year cycle of *yuga* with 1830 days in *Vedāṅga-Jyotiśa* (with antecedents in *Brāhmaṇa*s), both the Sun and the Moon were considered to complete *integral* numbers of revolutions (5 and 67 respectively) around the earth. In *Āryabhaṭīya* (499 CE) and subsequent texts, we have the notion of a *Mahāyuga* of 43,20,000 years in which apart from the Sun and the Moon, all five visible planets along with their apsides and nodes complete integral number of revolutions.[110] Again, the post-Vedic *Siddhānt*ic calendrical system with solar, sidereal, synodic and intercalary months, *nakṣatra*s and so on, is an advanced version of the *Vedāṅga-Jyotiśa* calendar, with the calculations based on the true positions ("longitudes") of the Sun and the Moon, rather than their mean positions as in the latter. Similarly, the procedure for finding the exact east directon, and the time from the shadow in the *Siddhānt*ic texts have their genesis in the *Śulbasūtra*s, *Arthaśāstra*, and the Jaina

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[107] A.K. Bag, op.cit.; K.V. Sarma, K. Ramasubramanian, M.D. Srinivas and M.S. Sriram, *Gaṇita-Yukti-Bhāṣā of Jyeṣṭhadeva.*

[108] See B. Datta and A.N. Singh (revised by K.S. Shukla), "Permutations and Combinations in India ", *Indian Journal of History of Science*, Vol.27(1), 1992, 231-244, for developments upto Bhāskara's times. For Nārāyaṇa's work on combinatorics, see Paramanand Singh, "*Gaṇitakaumudi* of Nārāyaṇa Paṇdita, Chapter XIII, English translation with notes", *Gaṇitabhāratī*, Vol. 23, 2001, 18-82.

[109] Raja Sridharan, R. Sridharan, and M.D. Srinivas, "Combinatorial Methods in Indian Music: *Pratyayas* in *Saṅgītaratnākara* of Śāraṅgadeva" in C.S. Seshadri, ed., *Studies in the History of Indian Mathematics*, 55-112.

[110] K.S. Shukla and K.V. Sarma, ed. with English translation and notes, *Āryabhaṭīya of Āryabhaṭa.*

and Buddhist works before 300 BCE.[111]

Indian mathematics and astronomy are algorithmic in nature.[112] The roots of this algorithmic approach are to found in the *sūtra* literature — *Śulbasūtra*s, *Chandaḥ-sūtra*, *Vedāṅga-Jyotiśa*, *Bhagavatīsūtra* and other works before 300 BCE.

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[111] See for instance, chapters 1 and 3 in K. Ramasubramanian and M.S. Sriram, *Tantrasaṅgraha of Nīlakaṇṭha Somayājī*.

[112] In fact, the word "algorithm" refers, etymologically, to the computational approach of ancient Indians. A treatise composed by Al-Khwārizmī around 820 CE expounded on the Indian decimal system and computational arithmetic. In Europe, Al-Khwārizmī's name got so closely associated with this new arithmetic that the Latin form of his name *algorismus* was given to any treatise on computational mathematics. Thus emerged the word "algorism", which later became "algorithm"!

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